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Mathematical modelling of linearly elastic shells

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The objective of this article is to lay down the proper *mathematical foundations of the two-dimensional theory of linearly elastic shells.* To this end, it provides, without any recourse to any *a priori* assumptions of a geometrical or mechanical nature, a *mathematical justification* of *two-dimensional linear shell theories*, by means of *asymptotic methods*, *with the thickness as the 'small' parameter.*

A major virtue of this approach is that it naturally leads to precise *mathematical definitions* of *linearly elastic 'membrane'* and *'flexural' shells.* Another noteworthy feature is that it highlights in particular the role played by *two fundamental tensors*, each associated with a displacement field of the middle surface, the *linearized change of metric* and *linearized change of curvature tensors.*

More specifically, under fundamentally distinct sets of assumptions bearing on the *geometry of the middle surface*, on the *boundary conditions*, and on the *order of magnitude of the applied forces*, it is shown that the three-dimensional displacements, once properly *scaled*, *converge* (in H^1, or in L^2, or in *ad hoc* completions) as the thickness approaches zero towards a 'two-dimensional' limit that satisfies either the *linear two-dimensional equations of a 'membrane' shell* (themselves divided into two subclasses) or the *linear two-dimensional equations of a 'flexural' shell.* Note that this *asymptotic analysis* automatically provides in each case the 'limit' *two-dimensional equations*, together with the *function space* over which they are well-posed.

The linear two-dimensional shell equations that are most commonly used in numerical simulations, namely *Koiter's equations*, *Naghdi's equations*, and *'shallow' shell equations*, are then carefully described, mathematically analysed, and likewise justified by means of asymptotic analyses.

The *existence* and *uniqueness* of solutions to each one of these linear two-dimensional shell equations are also established by means of crucial *inequalities of Korn's type on surfaces*, which are proved in detail at the beginning of the article.

This article serves as a mathematical basis for the numerically oriented companion article by Dominique Chapelle, also in this issue of *Acta Numerica.*

CONTENTS

1. Ubiquity of shells

A *shell* is a three-dimensional elastic body that is geometrically characterized by its middle surface and its 'small' thickness.

The *middle surface* S is a compact surface in $\mathbb{R}^3$ not contained in a plane (otherwise the shell is a plate) and it may or may not have a 'boundary' (for instance, the middle surface of a sail has a boundary, while that of a basketball has no boundary).

At each point $s \in S$, let $\boldsymbol{a}(s)$ denote a unit vector normal to S. Then the *reference configuration* of the shell, *i.e.*, the subset of $\mathbb{R}^3$ that it occupies 'before forces are applied to it', is a set of the form $\{(s + \zeta\boldsymbol{a}(s)) \in \mathbb{R}^3 : s \in S, |\zeta| \leq e(s)\}$, where the function $e : S \to \mathbb{R}$ is sufficiently smooth and satisfies $0 < e(s) \leq \varepsilon$ for all $s \in S$ and $\varepsilon > 0$ is thought of as being 'small' compared to some 'characteristic' length of S (its diameter for instance). If $e(s) = \varepsilon$ for all $s \in S$, the shell is said to have a *constant thickness* 2ε. If e is not a constant function, the shell is said to have a *variable thickness*.

Note that, since ε will essentially be used as a *dimensionless parameter* in the rest of this article, 2ε should thus be interpreted as the *ratio* between the actual thickness and a characteristic dimension of S, rather than as the thickness itself.

Shells and their assemblages constitute, or are found in, a wide variety of structures of considerable interest in contemporary engineering such as the blades of a rotor, an inner tube, a cooling tower, cylindrical tanks, balls used in various games, the sails and the hull of a sailing boat, a high-altitude scientific balloon (Figures 1.1 to 1.7); the doors, bumpers (fenders), bonnet (hood), windscreen (windshield), found in a car body; the wings, the tail, found in an aircraft; dams; parachutes.

Incidentally, these examples illustrate that actual shells generally have a variable thickness. For the sake of simplicity, we shall, however, only consider shells of constant thickness in this article, keeping in mind that this is not a serious restriction, as the effect of considering a variable thickness usually requires identical analyses, albeit involving substantially lengthier expressions at times.

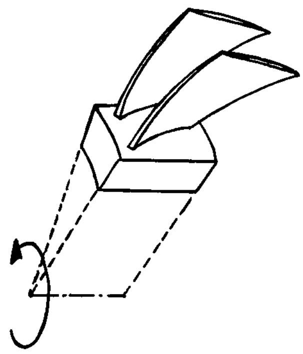

Fig. 1.1. A rotor and its blades provide an example of an elastic multi-structure, composed of a three-dimensional substructure (the rotor) and 'two-dimensional' substructures (the blades). Blades are often modelled as nonlinearly elastic 'shallow' shells

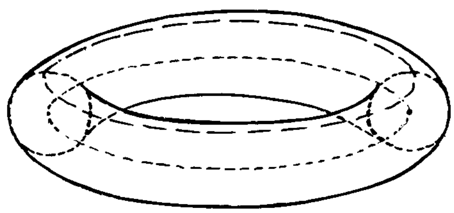

Fig. 1.2. An inner tube inside a tyre provides an example of a shell whose middle surface (a torus) has no boundary

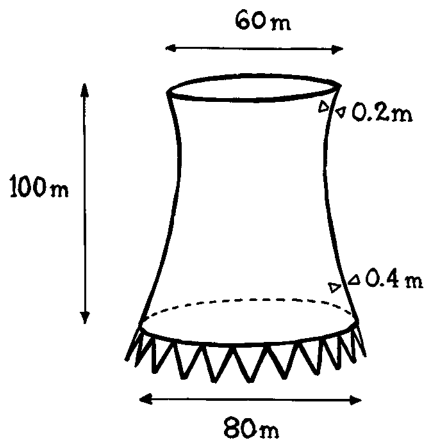

Fig. 1.3. A cooling tower in a utility plant: the middle surface is approximately a ruled hyperboloid of revolution; the height is of the order of 100 m, while the thickness varies from about 0.2 m at the top to about 0.4 m at the bottom, thus providing an instance of a ratio 2ε of approximately 1/500. Together with its supporting rods, a cooling tower constitutes another elastic multi-structure, composed of a 'two-dimensional' substructure (the shell) and 'one-dimensional' substructures (the rods). Although an instance of a *generalized membrane* shell (Section 9), such a shell is advantageously modelled by Koiter's equations (Section 11)

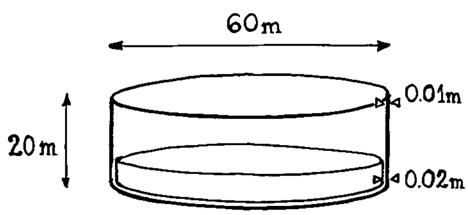

Fig. 1.4. A cylindrical tank for storing fuel in an oil refinery is another elastic multi-structure, composed of two 'two-dimensional' substructures: a cylindrical shell and a circular plate. In contemporary engineering, such a tank typically has a diameter of about 60 m, a height of 20 m, and a thickness varying from 0.04 m at the top to 0.02 m at the bottom, thus providing an instance where the ratio 2ε is approximately 1/5000

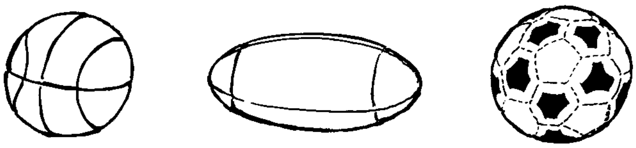

Fig. 1.5. Like an inner tube, balls used in various games provide examples of shells whose middle surface (a sphere or an ellipsoid-like surface) has no boundary. Another noteworthy feature, this time of a mechanical nature, of such shells is that they offer no resistance to crumpling when they are deflated. This observation alone suggests that they cannot be appropriately modelled by linear equations

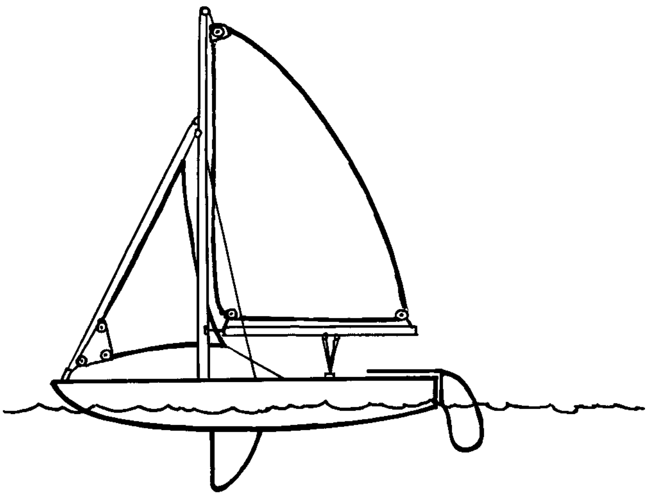

Fig. 1.6. The sails and the hull of a sailing boat provide two strikingly different instances of shells. Like a balloon, a sail offers no resistance unless it is already under tension (think of a spinnaker); thus it must also be modelled by nonlinear equations. By contrast, linear equations should suffice for the modelling of the hull, because it is not expected to undergo large displacements. But, even within the linear realm, the mathematical modelling of such a shell is an extremely challenging problem, for such a shell is usually made of 'composite', 'multi-layered' elastic materials

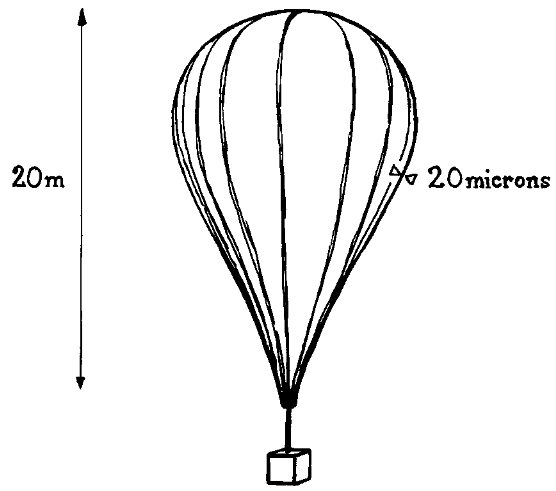

Fig. 1.7. A high-altitude scientific balloon provides a fascinating example. It is made by sealing together long, tapered, and originally flat sheets of polyethylene. The resulting structure is an incredibly thin shell, with an average thickness of about 20 microns and a height of about 20 m. The corresponding ratio 2ε is thus of the order of 10^{-6}, probably a world record! (This spectacular example was kindly brought to the author's attention by Frank Baginski, The George Washington University, Washington, DC.)

2. Why two-dimensional shell theories?

If any one of the structures described in Section 1 is viewed as a *three-dimensional elastic body*, the situation is on firm ground as regards its *mathematical modelling* (see, *e.g.*, Ciarlet (1988)). However, the situation is far from being idyllic as regards its *mathematical analysis*, at least if it is viewed as a *nonlinearly* elastic body. After the fundamental ideas set forth by Ball (1977) and his landmark existence result, there indeed remain various unresolved, and often exceedingly challenging, mathematical problems in nonlinear three-dimensional elasticity.

The *numerical analysis*, that is, the conception and mathematical analysis of convergent approximation schemes, most often finite element methods, is likewise well developed in three-dimensional elasticity, especially in the *linear* case (see in particular Ciarlet (1978, 1991), Glowinski (1984), Hughes (1987), Robert and Thomas (1991), Brezzi and Fortin (1991), Brenner and Scott (1994), Bathe (1996)), but also in the *nonlinear* case (see Le Tallec (1994) for an overview). There is nevertheless a strong proviso: three-dimensional numerical schemes almost invariably fail when they are applied to elastic structures that have a 'small' thickness, such as plates, shells, rods, and their assemblages.

The *'small' thickness* of a shell (or of a plate for that matter) makes it natural to 'replace' the genuine *three-dimensional model* by a 'simpler' *two-dimensional model*, that is, one that is posed over the middle surface of the shell. First, such a 'lower-dimensional' theory is of a *simpler mathematical structure*, which in turn generates a richer variety of results. Thus, while the 'global analysis', that is, the theories of existence, regularity, bifurcation, eversion phenomena, *etc.*, are still partly in their infancy in nonlinear *three-dimensional* elasticity (see in particular Marsden and Hughes (1983) and Ciarlet (1988)), such theories are by now on much firmer mathematical ground for the *two-dimensional* equations of nonlinearly elastic shells (see in particular Antman (1995) and Ciarlet (2000)).

In fact, not only is this replacement natural from a *theoretical* viewpoint, but it becomes a necessity when *numerical methods* must be devised for computing approximate displacements and stresses: any reasonably accurate three-dimensional discretization necessarily involves an astronomical number of unknowns, which renders it prohibitively expensive and makes its implementation extremely delicate, if not utterly impossible.

By contrast, the situation is on fairly safe ground, at least on the theoretical side, as regards the application of finite element methods to two-dimensional linear shell models: see in this respect Bernadou (1994) and the 'companion article' by Dominique Chapelle in this issue of *Acta Numerica.*

The above reasons clearly show why two-dimensional shell models are by and large preferred. Accordingly, three major questions naturally arise.

(i) How do we derive two-dimensional shell models in a systematic and rational manner from three-dimensional elasticity?

(ii) Has the mathematical analysis (existence, uniqueness, regularity, buckling, *etc.*, of solutions) of any known two-dimensional shell model reached a satisfactory stage?

(iii) In a given physical situation, how do we choose between the various 'available' two-dimensional shell models so that the chosen one be an 'as good as possible' approximation of the three-dimensional model it is supposed to 'replace'?

This last question is of paramount *practical* importance: it makes no sense to devise sophisticated numerical methods for accurately approximating the solution of the 'wrong' model!

The purpose of this article is to show how well-known, and sometimes not so well-known, two-dimensional linear shell equations can be fully justified by an asymptotic analysis of the three-dimensional equations, with the thickness as the 'small' parameter. It also provides a careful description of the physical situations where each kind of such equations should be safely employed.

This article thus only considers *linear* two-dimensional shell theories. A detailed justification from the same 'asymptotic' viewpoint, and a thorough mathematical analysis, of *nonlinear* two-dimensional shell theories are found in Ciarlet (2000).

Only recent references closely related to the 'asymptotic' approach followed here are listed in this article. The readers interested in an overview of the literature on shell theory may consult the reasonably complete bibliography provided, together with various historical commentaries, in Ciarlet (2000).

3. The three-dimensional Korn inequality in curvilinear coordinates

Although Sections 3 to 6 have a prelimininary character, they are essential: they provide an analysis of *Korn's inequalities in curvilinear coordinates*, whether in a three-dimensional domain or on a surface, which pervade most of the mathematical analysis of linearly elastic shells.

It is well known that the *three-dimensional Korn inequality* plays a fundamental role in establishing the *existence* and *uniqueness* of a solution in linearized three-dimensional elasticity in *Cartesian coordinates.* In essence, this inequality states that the $\mathbf{L}^2$-norm of the *linearized change of metric tensor* associated with displacement fields vanishing along a given portion, with area >0, of the boundary of a domain in $\mathbb{R}^3$, is equivalent to the $\mathbf{H}^1$-norm of these fields, represented by means of their Cartesian components.

The objective of this section is to show that the three-dimensional Korn inequality can in fact be directly established in *curvilinear coordinates*; *cf.* Theorem 3.4.

A *domain* Ω in $\mathbb{R}^n$ is an open, bounded, connected subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$, the set Ω being locally on one side of Γ. As Γ is Lipschitz-continuous, an *area element* $d\Gamma$ can be defined along Γ, and a *unit outer normal vector* $\boldsymbol{\nu} = (\nu_i)_{i=1}^n$ ('unit' meaning that its Euclidean norm is one) exists $d\Gamma$-almost everywhere along Γ.

Boldface letters denote vector-valued or matrix-valued functions and their associated function spaces. The norm in $L^2(\Omega)$ or $\mathbf{L}^2(\Omega)$ is denoted $|\cdot|_{0,\Omega}$ and the norm in the Sobolev spaces $H^m(\Omega)$ or $\mathbf{H}^m(\Omega), m \geq 1$, is denoted $\|\cdot\|_{m,\Omega}$. We also consider the Sobolev space

$$H^{-1}(\Omega) := \text{dual of space } H_0^1(\Omega).$$

It is clear that

$$v \in L^2(\Omega) \Rightarrow v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \leq i \leq n,$$

since (the duality between the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$)

$$\begin{aligned}|\langle v,\varphi\rangle| &= \left|\int_\Omega v\varphi\,\mathrm{d}x\right| \leq |v|_{0,\Omega}\|\varphi\|_{1,\Omega},\\ |\langle \partial_i v,\varphi\rangle| &= |-\langle v,\partial_i\varphi\rangle| = \left|-\int_\Omega v\partial_i\varphi\,\mathrm{d}x\right| \leq |v|_{0,\Omega}\|\varphi\|_{1,\Omega}\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Omega)$. It is *remarkable*, but also *remarkably difficult to prove*, that the converse implication holds.

Theorem 3.1: Lemma of J. L. Lions. Let Ω be a domain in $\mathbb{R}^n$ and let v be a distribution on Ω. Then

$$\left\{v \in H^{-1}(\Omega) \text{ and } \partial_i v \in H^{-1}(\Omega),\ 1 \leq i \leq n\right\} \Rightarrow v \in L^2(\Omega). \qquad \square$$

This implication was first proved by J. L. Lions, as stated in Magenes and Stampacchia (1958, p. 320, Note 27). Its first published proof for domains with smooth boundaries appeared in Duvaut and Lions (1972, p. 111); another proof was also given by Tartar (1978). Extensions to 'genuine' domains, that is, with Lipschitz-continuous boundaries, were then given by Bolley and Camus (1976), Geymonat and Suquet (1986), Borchers and Sohr (1990), and Amrouche and Girault (1994).

From now on, Latin indices or exponents take their values in the set $\{1,2,3\}$ (except if they are used for indexing sequences) and the summation convention is used. The Euclidean inner product and the vector product of two vectors $\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^3$ are denoted by $\boldsymbol{u}\cdot\boldsymbol{v}$ and $\boldsymbol{u}\wedge\boldsymbol{v}$; the Euclidean norm of $\boldsymbol{u} \in \mathbb{R}^3$ is denoted by $|\boldsymbol{u}|$.

Let Ω be a domain in $\mathbb{R}^3$, let $x = (x_i)$ denote a generic point in $\overline{\Omega}$, let $\partial_i = \partial/\partial x_i$, and let $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega};\mathbb{R}^3)$ be a $\mathcal{C}^1$-diffeomorphism such that the three vectors $\boldsymbol{g}_i(x) := \partial_i\boldsymbol{\Theta}(x)$ are linearly independent at all points $x \in \overline{\Omega}$. The three vectors $\boldsymbol{g}_i(x)$ form the *covariant basis* at the point $\boldsymbol{\Theta}(x)$, while the three vectors $\boldsymbol{g}^i(x)$ defined by the relations $\boldsymbol{g}^i(x)\cdot\boldsymbol{g}_j(x) = \delta^i_j$ form the *contravariant basis* at the same point (δ^i_j designates the Kronecker symbol).

In particular, the mapping $\boldsymbol{\Theta} : \overline{\Omega} \to \mathbb{R}^3$ is injective, so that any point $\widehat{x} \in \boldsymbol{\Theta}(\overline{\Omega})$ is the image of a well-defined point $x \in \overline{\Omega}$. The three coordinates x_i of x then constitute the *curvilinear coordinates* of $\widehat{x}$.

Let $g_{ij} := \boldsymbol{g}_i\cdot\boldsymbol{g}_j$ and $g^{ij} := \boldsymbol{g}^i\cdot\boldsymbol{g}^j$ denote the covariant and contravariant components of the *metric tensor* of the set $\{\widehat{\Omega}\}^-$, where $\widehat{\Omega} := \boldsymbol{\Theta}(\Omega)$, let $g := \det(g_{ij})$, so that $\sqrt{g}\,\mathrm{d}x$ denote the volume element in $\widehat{\Omega}$, and let $\Gamma^p_{ij} := \boldsymbol{g}^p\cdot\partial_i\boldsymbol{g}_j$ denote the *Christoffel symbols* (whenever no confusion should arise, the explicit dependence on $x \in \overline{\Omega}$ is henceforth omitted). The Christoffel symbols are used for computing the *first-order covariant derivatives*

$$v_{i\|j} := \partial_j v_i - \Gamma^p_{ij}v_p$$

of a vector field $v_i \boldsymbol{g}^i$ defined over the set $\overline{\Omega}$ (for details about these classical notions, see, *e.g.*, Ciarlet (2000, Section 1.2)).

Consider a homogeneous, isotropic, elastic body whose reference configuration is the set $\{\widehat{\Omega}\}^-$ and assume furthermore that $\{\widehat{\Omega}\}^-$ is a *natural state.* When the equations of three-dimensional elasticity are stated 'in curvilinear coordinates', that is, in terms of the coordinates of the set $\overline{\Omega} = \boldsymbol{\Theta}^{-1}(\{\widehat{\Omega}\}^-)$, the unknowns are the three covariant components $u_i : \overline{\Omega} \to \mathbb{R}$ of the displacement field $u_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ of the set $\{\widehat{\Omega}\}^-$. This means that, for each $x \in \overline{\Omega}$, $u_i(x)\boldsymbol{g}^i(x)$ is the displacement of the point $\boldsymbol{\Theta}(x) \in \boldsymbol{\Theta}(\overline{\Omega}) = \{\widehat{\Omega}\}^-$.

In particular, the variational equations of linearized three-dimensional elasticity in curvilinear coordinates take the following form (see, *e.g.*, Ciarlet (2000, Theorem 1.3-1)). The field $\boldsymbol{u} := (u_i)$ satisfies

$$\boldsymbol{u} \in \mathbf{V}(\Omega) := \left\{ \boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0 \right\},$$
$$\int_\Omega A^{ijkl} e_{k\|l}(\boldsymbol{u}) e_{i\|j}(\boldsymbol{v}) \sqrt{g}\, \mathrm{d}x = \int_\Omega f^i v_i \sqrt{g}\, \mathrm{d}x$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{V}(\Omega)$, where Γ_0 is a given subset of the boundary Γ of Ω with area $\Gamma_0 > 0$, the contravariant components of the *three-dimensional elasticity tensor* of the body are denoted by

$$A^{ijkl} := \lambda g^{ij} g^{kl} + \mu(g^{ik} g^{jl} + g^{il} g^{jk}),$$

the *Lamé constants* of the constituent elastic material are denoted by λ and μ, the covariant components of the *linearized change of metric tensor* associated with an arbitrary displacement field $v_i \boldsymbol{g}^i$ of the set $\{\widehat{\Omega}\}^-$ are denoted by

$$e_{i\|j}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p,$$

and the given functions $f^i \in L^2(\Omega)$ are the covariant components of the *applied body force* (we could as well consider surface forces acting on $\Gamma - \Gamma_0$). The functions $e_{i\|j}(\boldsymbol{v})$ are also called the *linearized strains in curvilinear coordinates.*

The interpretation of the functions $e_{i\|j}(\boldsymbol{v})$ is simple, yet crucial. Given an arbitrary displacement field $v_i \boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$ with sufficiently smooth covariant components $v_i : \overline{\Omega} \to \mathbb{R}$, let

$$g_{ij}(\boldsymbol{v}) := \partial_i(\boldsymbol{\Theta} + v_k \boldsymbol{g}^k) \cdot \partial_j(\boldsymbol{\Theta} + v_l \boldsymbol{g}^l)$$

denote the covariant components of the *metric tensor* of the *'deformed' set* $(\boldsymbol{\Theta} + v_i \boldsymbol{g}^i)(\overline{\Omega})$ associated with this displacement field. Then

$$e_{i\|j}(\boldsymbol{v}) = \frac{1}{2}[g_{ij}(\boldsymbol{v}) - g_{ij}]^{\text{lin}},$$

where $[\cdots]^{\text{lin}}$ denotes the linear part with respect to $\boldsymbol{v} = (v_i)$ in the expression $[\cdots]$ (for a proof, see Ciarlet (2000, Theorem 1.5-1)). The components

$e_{i\|j}(\boldsymbol{v})$ are thus aptly called those of the 'linearized', 'change of metric', tensor associated with the displacement field $v_i\boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$.

The boundary condition $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0, or the equivalent relation $u_i\boldsymbol{g}^i = \boldsymbol{0}$ on Γ_0, constitutes a (homogeneous) boundary condition of place. It states that the displacement field vanishes on the portion $\boldsymbol{\Theta}(\Gamma_0)$ of the boundary of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega}) = \{\widehat{\Omega}\}^-$.

Naturally, the usual equations of linearized three-dimensional elasticity in Cartesian coordinates are recovered by letting $\boldsymbol{\Theta} = \boldsymbol{id}$, in which case $g^{ij} = \delta^{ij}$, $\Gamma^p_{ij} = 0$, and $g = 1$.

Since there exists a constant $C_e = C_e(\Omega, \boldsymbol{\Theta}, \mu)$ such that

$$\sum_{i,j} |t_{ij}|^2 \leq C_e A^{ijkl}(x) t_{kl} t_{ij}$$

for all $x \in \overline{\Omega}$ and all symmetric matrices (t_{ij}) (see, *e.g.*, Ciarlet (2000, Theorem 1.8-1)), establishing the existence and uniqueness of a solution to the above variational problem thus amounts to establishing the existence of a constant C such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C \left\{ \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$ (all the other assumptions of the Lax–Milgram lemma are clearly satisfied). Our objective consists in proving that such a three-dimensional Korn inequality in curvilinear coordinates indeed holds (Theorem 3.4). Here, we follow Ciarlet (1993, 2000).

Such a Korn inequality is obtained in *three stages* (Theorems 3.2 to 3.4), the first one consisting in establishing, as a consequence of the Lemma of J. L. Lions (Theorem 3.1), a *Korn inequality valid for all vector fields* $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$, *i.e.*, that need not satisfy any boundary condition on Γ.

As its Cartesian special case, this inequality is truly remarkable, as only *six* different combinations of first-order partial derivatives, that is, $\frac{1}{2}(\partial_j v_i + \partial_i v_j)$, occur on its right-hand side, while all *nine* partial derivatives $\partial_j v_i$ occur on its left-hand side! A similarly striking observation applies to part (ii) of the proof of Theorem 3.2.

Theorem 3.2: Korn's inequality 'without boundary conditions' in curvilinear coordinates. Let Ω be a domain in $\mathbb{R}^3$ and let $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^3)$ be a $\mathcal{C}^1$-diffeomorphism of $\overline{\Omega}$ onto $\{\widehat{\Omega}\}^- = \boldsymbol{\Theta}(\overline{\Omega})$ such that the three vectors $\boldsymbol{g}_i = \partial_i\boldsymbol{\Theta}$ are linearly independent at all points of $\overline{\Omega}$. Given $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$, let

$$e_{i\|j}(\boldsymbol{v}) := \left\{ \frac{1}{2}(\partial_j v_i + \partial_i v_j) - \Gamma^p_{ij} v_p \right\} \in L^2(\Omega)$$

denote the covariant components of the linearized change of metric tensor associated with the displacement field $v_i \boldsymbol{g}^i$ of the set $\boldsymbol{\Theta}(\overline{\Omega})$. Then there exists a constant $C_0 = C_0(\Omega, \boldsymbol{\Theta})$ such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C_0 \left\{ \sum_i |v_i|^2_{0,\Omega} + \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2} \quad \text{for all } \boldsymbol{v} \in \mathbf{H}^1(\Omega).$$

Proof. The proof given here is essentially an extension of that given in Duvaut and Lions (1972, p. 110) for proving Korn's inequality without boundary conditions in Cartesian coordinates.

(i) Define the space

$$\mathbf{W}(\Omega) := \left\{ \boldsymbol{v} = (v_i) \in \mathbf{L}^2(\Omega) : e_{i\|j}(\boldsymbol{v}) \in L^2(\Omega) \right\}.$$

Then, $\mathbf{W}(\Omega)$ is a Hilbert space when equipped with the norm $\|\cdot\|_{\mathbf{W}(\Omega)}$ defined by

$$\|\boldsymbol{v}\|_{\mathbf{W}(\Omega)} := \left\{ \sum_i |v_i|^2_{0,\Omega} + \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}.$$

Note that the relations '$e_{i\|j}(\boldsymbol{v}) \in L^2(\Omega)$' are understood in the sense of distributions. They mean that there exist functions $e_{i\|j}(\boldsymbol{v})$ in $L^2(\Omega)$ such that

$$\int_\Omega e_{i\|j}(\boldsymbol{v}) \varphi \, \mathrm{d}x = -\int_\Omega \left\{ \frac{1}{2}(v_i \partial_j \varphi + v_j \partial_i \varphi) + \Gamma^p_{ij} v_p \varphi \right\} \mathrm{d}x \text{ for all } \varphi \in \mathcal{D}(\Omega).$$

Consider a Cauchy sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ with elements $\boldsymbol{v}^k = (v_i^k) \in \mathbf{W}(\Omega)$. By definition of the norm $\|\cdot\|_{\mathbf{W}(\Omega)}$, there exist functions $v_i \in L^2(\Omega)$ and $e_{i\|j} \in L^2(\Omega)$ such that

$$v_i^k \to v_i \text{ in } L^2(\Omega) \text{ and } e_{i\|j}(\boldsymbol{v}^k) \to e_{i\|j} \text{ in } L^2(\Omega) \text{ as } k \to \infty,$$

since the space $L^2(\Omega)$ is complete. Given a function $\varphi \in \mathcal{D}(\Omega)$, letting $k \to \infty$ in the relations

$$\int_\Omega e_{i\|j}(\boldsymbol{v}^k) \varphi \, \mathrm{d}x = -\int_\Omega \left\{ \frac{1}{2}(v_i^k \partial_j \varphi + v_j^k \partial_i \varphi) + \Gamma^p_{ij} v_p^k \varphi \right\} \mathrm{d}x, \ k \geq 1,$$

shows that $e_{i\|j} = e_{i\|j}(\boldsymbol{v})$.

(ii) The spaces $\mathbf{W}(\Omega)$ and $\mathbf{H}^1(\Omega)$ coincide.

Clearly, $\mathbf{H}^1(\Omega) \subset \mathbf{W}(\Omega)$. To prove the other inclusion, let $\boldsymbol{v} = (v_i) \in \mathbf{W}(\Omega)$. Then

$$e_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_j v_i + \partial_i v_j) = \left\{ e_{i\|j}(\boldsymbol{v}) + \Gamma^p_{ij} v_p \right\} \in L^2(\Omega),$$

since $e_{i\|j}(\boldsymbol{v}) \in L^2(\Omega), \Gamma_{ij}^p \in \mathcal{C}^0(\overline{\Omega})$, and $v_p \in L^2(\Omega)$. We thus have

$$\partial_k v_i \in H^{-1}(\Omega),$$
$$\partial_j(\partial_k v_i) = \{\partial_j e_{ik}(\boldsymbol{v}) + \partial_k e_{ij}(\boldsymbol{v}) - \partial_i e_{jk}(\boldsymbol{v})\} \in H^{-1}(\Omega),$$

since $w \in L^2(\Omega)$ implies $\partial_k w \in H^{-1}(\Omega)$. Hence $\partial_k v_i \in L^2(\Omega)$ by the Lemma of J. L. Lions (Theorem 3.1) and thus $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$.

(iii) Korn's inequality without boundary conditions.

The identity mapping ι from the space $\mathbf{H}^1(\Omega)$ equipped with $\|\cdot\|_{1,\Omega}$ into the space $\mathbf{W}(\Omega)$ equipped with $\|\cdot\|_{\mathbf{W}(\Omega)}$ is injective, continuous (there clearly exists a constant c such that $\|\boldsymbol{v}\|_{\mathbf{W}(\Omega)} \leq c\|\boldsymbol{v}\|_{1,\Omega}$ for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega)$), and surjective by (ii). Since both spaces are complete (*cf.* (i)), the closed graph theorem then shows that the inverse mapping ι^{-1} is also continuous. This continuity is exactly what Korn's inequality without boundary conditions states. □

Our next objective is to 'get rid' of the norms $|v_i|_{0,\Omega}$ on the right-hand side of the Korn inequality established in Theorem 3.2 when the fields $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ are subjected to the boundary condition $\boldsymbol{v} = \mathbf{0}$ on $\Gamma_0 \subset \Gamma$ and area $\Gamma_0 > 0$. As a preliminary, we establish the weaker property that the *seminorm* $\boldsymbol{v} \to \{\sum_{i,j} |e_{i\|j}(\boldsymbol{v})|_{0,\Omega}^2\}^{1/2}$ becomes a *norm* for such fields, by generalizing to curvilinear coordinates the well-known *linearized rigid displacement lemma in Cartesian coordinates*. 'Linearized' reminds us that if $e_{i\|j}(\boldsymbol{v}) = 0$ in Ω, that is, if only the *linearized* part of the change of metric tensor vanishes, the corresponding displacement field $v_i\boldsymbol{g}^i$ is likewise only the *linearized approximation* to a genuine rigid displacement.

Part (a) in the next theorem is a linearized rigid displacement lemma *without boundary conditions*, while part (b) is a linearized rigid displacement lemma *with boundary conditions*.

Theorem 3.3: Linearized rigid displacement lemma in curvilinear coordinates. Let the assumptions be as in Theorem 3.2.

(a) Let $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ be such that

$$e_{i\|j}(\boldsymbol{v}) = 0 \text{ in } \Omega.$$

Then the vector field $v_i\boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ is a 'linearized rigid displacement' of the set $\boldsymbol{\Theta}(\overline{\Omega})$, in the sense that there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$v_i(x)\boldsymbol{g}^i(x) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \boldsymbol{\Theta}(x) \text{ for all } x \in \overline{\Omega}.$$

(b) Let Γ_0 be a $d\Gamma$-measurable subset of $\Gamma = \partial\Omega$ that satisfies

$$\text{area } \Gamma_0 > 0.$$

Then

$$\left.\begin{aligned} \boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega), \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0, \\ e_{i\|j}(\boldsymbol{v}) = 0 \text{ in } \Omega \end{aligned}\right\} \Rightarrow \boldsymbol{v} = \mathbf{0} \text{ in } \Omega.$$

Proof. Let $\widehat{\boldsymbol{e}}_i = \widehat{\boldsymbol{e}}^i$ denote the basis vectors of the Cartesian frame. It is verified that the following relations hold:

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}})(\widehat{x}) = \big(e_{k\|l}(\boldsymbol{v})[\boldsymbol{g}^k]_i[\boldsymbol{g}^l]_j\big)(x) \text{ for all } \widehat{x} = (\widehat{x}_i) := \boldsymbol{\Theta}(x), x \in \Omega,$$

where $\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) := \frac{1}{2}(\widehat{\partial}_j\widehat{v}_i + \widehat{\partial}_i\widehat{v}_j), \widehat{\partial}_i := \partial/\partial\widehat{x}_i, [\boldsymbol{g}^k]_i := \boldsymbol{g}^k \cdot \widehat{\boldsymbol{e}}_i$ denote the ith Cartesian component of the vector $\boldsymbol{g}^k$, and the vector fields $\widehat{\boldsymbol{v}} = (\widehat{v}_i) \in \mathbf{H}^1(\widehat{\Omega})$ and $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ are related by

$$\widehat{v}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = v_i(x)\boldsymbol{g}^i(x) \text{ for all } \widehat{x} = \boldsymbol{\Theta}(x),\ x \in \Omega.$$

Hence

$$e_{i\|j}(\boldsymbol{v}) = 0 \text{ in } \Omega \Rightarrow \widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0 \text{ in } \widehat{\Omega},$$

and the identity (the same as in the proof of Theorem 3.2)

$$\widehat{\partial}_j(\widehat{\partial}_k\widehat{v}_i) = \widehat{\partial}_j\widehat{e}_{ik}(\widehat{\boldsymbol{v}}) + \widehat{\partial}_k\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) - \widehat{\partial}_i\widehat{e}_{jk}(\widehat{\boldsymbol{v}}) \text{ in } \mathcal{D}'(\widehat{\Omega})$$

further shows that

$$\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0 \text{ in } \widehat{\Omega} \Rightarrow \widehat{\partial}_j(\widehat{\partial}_k\widehat{v}_i) = 0 \text{ in } \mathcal{D}'(\widehat{\Omega}).$$

By a classical result from distribution theory (Schwartz 1966, p. 60), each function $\widehat{v}_i$ is therefore a polynomial of degree ≤ 1 (the set $\widehat{\Omega}$ is connected). In other words, there exist constants $\widehat{c}_i$ and $\widehat{d}_{ij}$ such that

$$\widehat{v}_i(\widehat{x}) = \widehat{c}_i + \widehat{d}_{ij}\widehat{x}_j \text{ for all } \widehat{x} = (\widehat{x}_i) \in \widehat{\Omega}.$$

But $\widehat{e}_{ij}(\widehat{\boldsymbol{v}}) = 0$ also implies that $\widehat{d}_{ij} = -\widehat{d}_{ji}$; hence there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$\widehat{\boldsymbol{v}}_i(\widehat{x})\widehat{\boldsymbol{e}}^i = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \widehat{\boldsymbol{O}}\widehat{\boldsymbol{x}} \text{ for all } \widehat{x} \in \widehat{\Omega},$$

and hence such that

$$v_i(x)\boldsymbol{g}^i(x) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \boldsymbol{\Theta}(x) \text{ for all } x \in \Omega.$$

Since the set where such a vector field $\widehat{v}_i\widehat{\boldsymbol{e}}^i$ vanishes is always of zero area unless $\widehat{\boldsymbol{c}} = \widehat{\boldsymbol{d}} = \mathbf{0}$ (as is easily proved), it follows that $\widehat{\boldsymbol{v}} = \mathbf{0}$ in $\widehat{\Omega}$, and hence that $\boldsymbol{v} = \mathbf{0}$ in Ω, when area $\Gamma_0 > 0$. □

We are now in a position to prove a fundamental inequality in curvilinear coordinates.

Theorem 3.4: Three-dimensional Korn's inequality in curvilinear coordinates. Let the assumptions be as in Theorem 3.2, let Γ_0 be a $d\Gamma$-measurable subset of $\Gamma = \partial\Omega$ that satisfies

$$\text{area } \Gamma_0 > 0,$$

and let the space $\mathbf{V}(\Omega)$ be defined by

$$\mathbf{V}(\Omega) := \left\{ \boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0 \right\}.$$

Then there exists a constant $C = C(\Omega, \Gamma_0, \boldsymbol{\Theta})$ such that

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C \left\{ \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2} \text{ for all } \boldsymbol{v} \in \mathbf{V}(\Omega).$$

Proof. Given $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$, let

$$|\boldsymbol{v}|_{\mathbf{W}(\Omega)} := \left\{ \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}.$$

If the stated inequality is false, then there exists a sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ of elements $\boldsymbol{v}^k \in \mathbf{V}(\Omega)$ such that

$$\|\boldsymbol{v}^k\|_{1,\Omega} = 1 \text{ for all } k \text{ and } \lim_{k\to\infty} |\boldsymbol{v}^k|_{\mathbf{W}(\Omega)} = 0.$$

Since the sequence $(\boldsymbol{v}^k)_{k=1}^\infty$ is bounded in $\mathbf{H}^1(\Omega)$, there exists a subsequence $(\boldsymbol{v}^l)_{l=1}^\infty$ that converges in $\mathbf{L}^2(\Omega)$ by the Rellich–Kondrašov theorem; furthermore, since $\lim_{l\to\infty} |\boldsymbol{v}^l|_{\mathbf{W}(\Omega)} = 0$, each sequence $(e_{i\|j}(\boldsymbol{v}^l))_{l=1}^\infty$ also converges in $L^2(\Omega)$ (to 0, but this information is not used at this stage). The subsequence $(\boldsymbol{v}^l)_{l=1}^\infty$ is thus a Cauchy sequence with respect to the norm

$$\boldsymbol{v} = (v_i) \to \left\{ \sum_i |v_i|^2_{0,\Omega} + \sum_{i,j} |e_{i\|j}(\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2},$$

and hence with respect to the norm $\|\cdot\|_{1,\Omega}$ by Korn's inequality without boundary conditions (Theorem 3.2).

The space $\mathbf{V}(\Omega)$ is complete, being a closed subspace of $\mathbf{H}^1(\Omega)$; thus there exists $\boldsymbol{v} \in \mathbf{V}(\Omega)$ such that

$$\boldsymbol{v}^l \to \boldsymbol{v} \text{ in } \mathbf{H}^1(\Omega),$$

and the limit $\boldsymbol{v}$ satisfies $|e_{i\|j}(\boldsymbol{v})|_{0,\Omega} = \lim_{l\to\infty} |e_{i\|j}(\boldsymbol{v}^l)|_{0,\Omega} = 0$; hence $\boldsymbol{v} = \mathbf{0}$ by Theorem 3.3. But this contradicts the relations $\|\boldsymbol{v}^l\|_{1,\Omega} = 1$ for all $l \geq 1$, and the proof is complete. □

Letting $\mathbf{\Theta} = \boldsymbol{id}$ shows that Theorems 3.2, 3.3, and 3.4 contain as special cases the Korn inequalities and the linearized rigid displacement lemma in Cartesian coordinates (see, *e.g.*, Duvaut and Lions (1972)).

4. Inequality of Korn's type on a general surface

The theory of linearly elastic shells leads to 'two-dimensional' models, *i.e.*, that are defined in terms of *curvilinear coordinates* of the middle surface of the shell. The objective of Sections 4 to 6 is to show that *inequalities of Korn's type on a surface* can be established in terms of its curvilinear coordinates. As we shall see, such inequalities play a fundamental role in establishing the *existence* and *uniqueness* of solutions to such two-dimensional shell equations as the *Koiter*, *flexural*, and *membrane* ones. They also play a crucial role in the asymptotic analysis of the three-dimensional equations that justifies such two-dimensional models.

While a three-dimensional domain in $\mathbb{R}^3$ is unambiguously defined by a *single* tensor field, the metric tensor field (up to rigid deformations, of course) of a surface instead requires *two* tensor fields for its definition: the *metric tensor* field again and in addition the *curvature tensor* field, also called the *first* and *second fundamental forms* of the surface.

An inequality of Korn's type on a *general* surface can then be established. In essence, it states that, for a general surface S, the $\mathbf{L}^2$-norm of the *linearized change of metric tensor*, plus the $\mathbf{L}^2$-norm of the *linearized change of curvature tensor* (associated with displacement fields of S vanishing together with the normal derivative of their normal component along a given portion, with length > 0, of the 'boundary' of S) is equivalent to the $(H^1 \times H^1 \times H^2)$-norm of these fields, expressed here in curvilinear coordinates (both tangential components of the displacement fields are in H^1 and their normal components are in H^2); *cf.* Theorem 4.4.

To begin with, we briefly recall some basic results on the differential geometry of surfaces in $\mathbb{R}^3$; for references, see, *e.g.*, Stoker (1969), Klingenberg (1973), do Carmo (1976), Berger and Gostiaux (1992), Sanchez-Hubert and Sanchez-Palencia (1997), or Ciarlet (2000, Sections 2.1 to 2.5). Latin indices or components vary as before in the set $\{1, 2, 3\}$; in addition, Greek indices (except ν in ∂_ν) or exponents (except ε) vary in the set $\{1, 2\}$, and the summation convention now applies to both kinds of indices and exponents.

Let ω be a two-dimensional domain with boundary γ, let $y = (y_\alpha)$ denote a generic point in $\overline{\omega}$, let $\partial_\alpha = \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} = \partial^2/\partial y_\alpha \partial y_\beta$, and let an injective mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be given such that the two vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \overline{\omega}$. They then form the *covariant basis of the tangent plane* to the surface $S := \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$, while the two vectors $\boldsymbol{a}^\alpha(y)$ of the tangent plane defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$ form the *contravariant basis of the tangent* *plane* at $\boldsymbol{\theta}(y)$ (δ^α_β designates the Kronecker symbol). Let

$$\boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|};$$

then the vectors $\boldsymbol{a}^i(y)$ form the contravariant basis at the point $\boldsymbol{\theta}(y) \in S$.

The mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ being in particular injective, any point $\widehat{y}$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is the image of a well-defined point y in the set $\overline{\omega}$. The two coordinates y_α of y then constitute the *curvilinear coordinates* of $\widehat{y}$.

The *metric tensor*, or *first fundamental form*, of the surface S is defined by its *covariant components*

$$a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta = a_{\beta\alpha},$$

or by its *contravariant components*

$$a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta = a^{\beta\alpha}$$

(we omit the explicit dependence on $y \in \overline{\omega}$ when no confusion should arise). Note that the determinant

$$a := \det(a_{\alpha\beta})$$

is everywhere > 0 in $\overline{\omega}$ since the symmetric matrix $(a_{\alpha\beta})$ is positive definite in $\overline{\omega}$. The area element along S is $\sqrt{a}\,\mathrm{d}y$.

The *curvature tensor*, or *second fundamental form*, of S is defined by its *covariant components*

$$b_{\alpha\beta} := \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}_\beta = -\partial_\alpha \boldsymbol{a}_3 \cdot \boldsymbol{a}_\beta = b_{\beta\alpha},$$

or by its *mixed components*

$$b^\beta_\alpha := a^{\beta\sigma} b_{\sigma\alpha}.$$

The *Christoffel symbols*

$$\Gamma^\sigma_{\alpha\beta} := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\beta\alpha}$$

are used for computing the functions

$$\eta_{\beta|\alpha} := \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma \quad \text{ and } \quad \eta_{3|\alpha\beta} := \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3,$$

which are instances of first-order and second-order covariant derivatives of a vector field $\eta_i \boldsymbol{a}^i$ defined over the surface S, or for computing the functions

$$b^\tau_\beta|_\alpha := \partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma} b^\sigma_\beta + \Gamma^\sigma_{\alpha\beta} b^\tau_\sigma,$$

which are instances of first-order covariant derivatives of the curvature tensor of S, defined here by means of its mixed components.

The two-dimensional Koiter equations for a linearly elastic shell, which have been proposed by Koiter (1970), take the following form. The *unknowns* are the covariant components $\zeta^\varepsilon_{i,K} : \overline{\omega} \to \mathbb{R}$ of the displacement field

$\zeta^\varepsilon_{i,K}\boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell; $\boldsymbol{\zeta}^\varepsilon_K := (\zeta^\varepsilon_{i,K})$ satisfies

$$\boldsymbol{\zeta}^\varepsilon_K \in \mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y \\ = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

(∂_ν denoting the outer normal derivative operator along γ) for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$; γ_0 is a subset of γ with length $\gamma_0 > 0$; $2\varepsilon > 0$ is the thickness of the shell;

$$a^{\alpha\beta\sigma\tau,\varepsilon} := \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon}a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

denote the contravariant components of the two-dimensional elasticity tensor of the shell, λ^ε and μ^ε being the Lamé constants of the elastic material constituting the shell; the given functions $p^{i,\varepsilon} \in L^2(\omega)$ account for the applied forces. Finally, $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ denote the covariant components of the linearized change of metric and linearized change of curvature tensors of S:

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3 \\
&= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \\
\rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \eta_{3|\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha\eta_{\sigma|\beta} + b^\tau_\beta\eta_{\tau|\alpha} + b^\tau_\beta|_\alpha\eta_\tau \\
&= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 \\
&\quad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) \\
&\quad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau.
\end{aligned}$$

These functions play a fundamental role in linearized shell theory. As we shall see, they systematically appear in the linear two-dimensional shell equations later justified in this article!

Their interpretation, which is thus crucial for the understanding of these equations, is the following. Given an arbitrary displacement field $\eta_i\boldsymbol{a}^i$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$ with sufficiently smooth covariant components $\eta_i : \overline{\omega} \to \mathbb{R}$, let

$$a_\alpha(\boldsymbol{\eta}) := \partial_\alpha(\boldsymbol{\theta} + \eta_i\boldsymbol{a}^i) \text{ and } \boldsymbol{a}_3(\boldsymbol{\eta}) := \frac{\boldsymbol{a}_1(\boldsymbol{\eta}) \wedge \boldsymbol{a}_2(\boldsymbol{\eta})}{|\boldsymbol{a}_1(\boldsymbol{\eta}) \wedge \boldsymbol{a}_2(\boldsymbol{\eta})|}$$

denote the vectors of the covariant bases attached to the 'deformed' surface $(\boldsymbol{\theta} + \eta_i\boldsymbol{a}^i)(\overline{\omega})$ associated with this displacement field. Since the vectors

$\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent in $\overline{\omega}$ by assumption, so are the vectors $\boldsymbol{a}_\alpha(\boldsymbol{\eta})$ provided the fields $\boldsymbol{\eta} = (\eta_i)$ are sufficiently small (*e.g.*, with respect to the norm of the space $\mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$); hence the vector $\boldsymbol{a}_3(\boldsymbol{\eta})$ is well defined for such fields. The following interpretation is thus legitimate, because it only pertains to the linearized theory 'around $\boldsymbol{\eta} = \mathbf{0}$'.

Let

$$a_{\alpha\beta}(\boldsymbol{\eta}) := \boldsymbol{a}_\alpha(\boldsymbol{\eta}) \cdot \boldsymbol{a}_\beta(\boldsymbol{\eta}) \quad \text{and} \quad b_{\alpha\beta}(\boldsymbol{\eta}) := \boldsymbol{a}_3(\boldsymbol{\eta}) \cdot \partial_\alpha \boldsymbol{a}_\beta(\boldsymbol{\eta})$$

denote the covariant components of the metric and curvature tensors of the deformed surface $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$. Then

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}\left[a_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}\right]^{\text{lin}},$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = \left[b_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}\right]^{\text{lin}},$$

where $[\cdots]^{\text{lin}}$ denotes the linear part with respect to $\boldsymbol{\eta} = (\eta_i)$ in the expression $[\cdots]$ (for a proof, see Ciarlet (2000, Theorems 2.4-1 and 2.5-1)). The components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are thus aptly called those of the 'linearized', 'change of metric' and 'change of curvature' tensors associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the surface S.

Koiter's equations are of paramount importance in engineering practice, as they are very often used in numerical simulations of shell structures. They are further studied, and in particular fully justified, in Section 11.

As is easily seen (see *e.g.* Bernadou, Ciarlet and Miara (1994, Lemma 2.1), or Ciarlet (2000, Theorem 3.3-2)), there exists a constant $c_e = c_e(\omega, \boldsymbol{\theta}, \mu^\varepsilon)$ such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \le c_e a^{\alpha\beta\sigma\tau,\varepsilon}(y) t_{\sigma\tau} t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ and there exists a constant a_0 such that $a(y) \ge a_0 > 0$ for all $y \in \overline{\omega}$. Establishing the existence and uniqueness of a solution to this variational problem by the Lax–Milgram lemma thus amounts to establishing the existence of a constant c such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega} \right\}^{1/2} \le c \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$.

The objective of this section consists in showing that such an inequality of Korn's type indeed holds for a general surface (Theorem 4.4).

As is readily checked, the same inequality of Korn's type on a surface also provides an existence and uniqueness theorem for the two-dimensional equations of a linearly elastic 'flexural' shell. These equations, which will be fully justified in Section 10 through an asymptotic analysis of the three-dimensional solutions under the assumption that the space

$$\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} \in \mathbf{V}_K(\omega) : \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

does not reduce to $\{\mathbf{0}\}$, consist in finding the solution $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$ of the following variational problem:

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_F(\omega),$$

$$\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$.

In Section 3, we established 'three-dimensional' Korn inequalities, first without (Theorem 3.2), then with (Theorem 3.4), boundary conditions (the second one depending on a three-dimensional linearized rigid displacement lemma; *cf.* Theorem 3.3). Both inequalities involved the covariant components $e_{i\|j}(\boldsymbol{v})$ of the three-dimensional linearized change of metric tensor.

But while only *one* tensor, the metric tensor, is attached to a three-dimensional domain in $\mathbb{R}^3$, *two* tensors, the metric and curvature tensors, are attached to a surface in $\mathbb{R}^3$. It is thus natural to likewise establish inequalities of Korn's type *on a surface*, first *without* (Theorem 4.1), then *with* (Theorem 4.4), boundary conditions (the second one again depending on a linearized rigid displacement lemma, this time on a surface; *cf.* Theorem 4.3), such inequalities now involving the covariant components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of both its linearized change of metric tensor and linearized change of curvature tensor.

We shall establish that these inequalities are valid for a 'general' surface $S = \boldsymbol{\theta}(\overline{\omega})$, that is, corresponding to a general mapping $\boldsymbol{\theta}$ (except that $\boldsymbol{\theta}$ should be sufficiently smooth; 'less smooth' mappings $\boldsymbol{\theta}$ are considered in Section 5). In other words, no restriction is imposed on the 'geometry' of S (in contrast, such a restriction holds for the inequality of Korn's type that will be established in Section 6).

The linearized rigid displacement lemma (Theorem 4.3) and the inequality of Korn's type on a general surface (Theorem 4.4) were first established by Bernadou and Ciarlet (1976). A simpler presentation, which we follow here, was then proposed by Ciarlet and Miara (1992*b*) (see also Bernadou, Ciarlet and Miara (1994)). Its first stage consists in establishing an inequality of Korn's type 'without boundary conditions', again as a consequence of the Lemma of J. L. Lions (as in dimension three; *cf.* Theorem 3.2).

Theorem 4.1: Inequality of Korn's type 'without boundary conditions' on a general surface. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$. Given $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times$

$H^1(\omega) \times H^2(\omega)$, let

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \left\{\frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3\right\} \in L^2(\omega),\\
\rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \big\{\partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3\\
&\qquad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta\,(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma)\\
&\qquad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau\big\} \in L^2(\omega)
\end{aligned}$$

denote the covariant components of the linearized change of metric and linearized change of curvature tensors associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the surface $\boldsymbol{\theta}(\overline{\omega})$. Then there exists a constant $c_0 = c_0(\omega, \boldsymbol{\theta})$ such that

$$\begin{aligned}
&\left\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2}\\
&\qquad \le c_0 \left\{\sum_\alpha |\eta_\alpha|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta}|\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\}^{1/2}
\end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

Proof. **(i)** Define the space

$$\begin{aligned}
\mathbf{W}_K(\omega) \;=\; &\big\{\boldsymbol{\eta} = (\eta_i) \in L^2(\omega) \times L^2(\omega) \times H^1(\omega) :\\
&\qquad \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega), \rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)\big\}.
\end{aligned}$$

Then $\mathbf{W}_K(\omega)$ is a Hilbert space when equipped with the norm $\|\cdot\|^K_\omega$ defined by

$$\|\boldsymbol{\eta}\|^K_\omega := \left\{\sum_\alpha |\eta_\alpha|^2_{0,\omega} + \|\eta_3\|^2_{1,\omega} + \sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_{\alpha,\beta}|\rho_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\}^{1/2}.$$

The relations '$\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$' and '$\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$' appearing in the definition of the space $\mathbf{W}_K(\omega)$ are to be understood in the sense of distributions. They mean that $\boldsymbol{\eta} = (\eta_i) \in L^2(\omega) \times L^2(\omega) \times H^1(\omega)$ belongs to $\mathbf{W}_K(\omega)$ if there exist functions in $L^2(\omega)$, denoted by $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$, such that, for all $\varphi \in \mathcal{D}(\omega)$,

$$\begin{aligned}
\int_\omega \gamma_{\alpha\beta}(\boldsymbol{\eta})\varphi\,\mathrm{d}\omega \;&=\; -\int_\omega \left\{\frac{1}{2}(\eta_\beta\partial_\alpha\varphi + \eta_\alpha\partial_\beta\varphi) + \Gamma^\sigma_{\alpha\beta}\eta_\sigma\varphi + b_{\alpha\beta}\eta_3\varphi\right\}\mathrm{d}\omega,\\
\int_\omega \rho_{\alpha\beta}(\boldsymbol{\eta})\varphi\,\mathrm{d}\omega \;&=\; -\int_\omega \big\{\partial_\alpha\eta_3\partial_\beta\varphi + \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3\varphi + b^\sigma_\alpha b_{\sigma\beta}\eta_3\varphi\\
&\qquad + b^\sigma_\alpha(\eta_\sigma\partial_\beta\varphi + \Gamma^\tau_{\beta\sigma}\eta_\tau\varphi) + b^\tau_\beta(\eta_\tau\partial_\alpha\varphi + \Gamma^\sigma_{\alpha\tau}\eta_\sigma\varphi)\\
&\qquad + \big(\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma\big)\eta_\tau\varphi\big\}\,\mathrm{d}\omega.
\end{aligned}$$

Consider a Cauchy sequence $(\boldsymbol{\eta}^k)_{k=1}^{\infty}$ with elements $\boldsymbol{\eta}^k = (\eta_i^k) \in \mathbf{W}_K(\omega)$. The definition of the norm $\|\cdot\|_\omega^K$ shows that there exist $\eta_\alpha \in L^2(\omega), \eta_3 \in H^1(\omega), \gamma_{\alpha\beta} \in L^2(\omega)$, and $\rho_{\alpha\beta} \in L^2(\omega)$ such that

$$\begin{aligned} \eta_\alpha^k &\to \eta_\alpha \text{ in } L^2(\omega), & \eta_3^k &\to \eta_3 \text{ in } H^1(\omega), \\ \gamma_{\alpha\beta}(\boldsymbol{\eta}^k) &\to \gamma_{\alpha\beta} \text{ in } L^2(\omega), & \rho_{\alpha\beta}(\boldsymbol{\eta}^k) &\to \rho_{\alpha\beta} \text{ in } L^2(\omega) \end{aligned}$$

as $k \to \infty$. Given a function $\varphi \in \mathcal{D}(\omega)$, letting $k \to \infty$ in the relations $\int_\omega \gamma_{\alpha\beta}(\boldsymbol{\eta}^k)\varphi \,\mathrm{d}\omega = \cdots$ and $\int_\omega \rho_{\alpha\beta}(\boldsymbol{\eta}^k)\varphi \,\mathrm{d}\omega = \cdots$ then shows that $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta})$.

(ii) The spaces $\mathbf{W}_K(\omega)$ and $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ coincide.

Clearly, $H^1(\omega) \times H^1(\omega) \times H^2(\omega) \subset \mathbf{W}_K(\omega)$. To prove the other inclusion, let $\boldsymbol{\eta} = (\eta_i) \in \mathbf{W}_K(\omega)$. The relations

$$e_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) = \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \Gamma_{\alpha\beta}^\sigma \eta_\sigma + b_{\alpha\beta}\eta_3$$

then imply that $e_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ since the functions $\Gamma_{\alpha\beta}^\sigma$ and $b_{\alpha\beta}$ are continuous on $\overline{\omega}$ (in fact, even continuously differentiable; recall that we assume $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$). Therefore

$$\begin{aligned} \partial_\sigma \eta_\alpha &\in H^{-1}(\omega), \\ \partial_\beta(\partial_\sigma \eta_\alpha) = \{\partial_\beta e_{\alpha\sigma}(\boldsymbol{\eta}) + \partial_\sigma e_{\alpha\beta}(\boldsymbol{\eta}) - \partial_\alpha e_{\beta\sigma}(\boldsymbol{\eta})\} &\in H^{-1}(\omega), \end{aligned}$$

since $\theta \in L^2(\omega)$ implies $\partial_\sigma \theta \in H^{-1}(\omega)$. Hence $\partial_\sigma \eta_\sigma \in L^2(\omega)$ by the Lemma of J. L. Lions (Theorem 3.1), and thus $\eta_\alpha \in H^1(\omega)$.

The definition of the functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$, the continuity over $\overline{\omega}$ of the functions $\Gamma_{\alpha\beta}^\sigma, b_{\sigma\beta}, b_\alpha^\sigma$, and $\partial_\alpha b_\beta^\tau$, and the relations $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ then imply that $\partial_{\alpha\beta}\eta_3 \in L^2(\omega)$, and hence that $\eta_3 \in H^2(\omega)$.

(iii) Inequality of Korn's type without boundary conditions.

The identity mapping ι from the space $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ equipped with its product norm $\boldsymbol{\eta} = (\eta_i) \to \{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2\}^{1/2}$ into the space $\mathbf{W}_K(\omega)$ equipped with $\|\cdot\|_\omega^K$ is injective, continuous, and surjective by (ii). Since both spaces are complete (*cf.* (i)), the closed graph theorem then shows that the inverse mapping ι^{-1} is also continuous or, equivalently, that the inequality of Korn's type without boundary conditions holds. $\square$

In order to establish an inequality of Korn's type 'with boundary conditions', we have to identify classes of boundary conditions to be imposed on the fields $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ in order that we can 'get rid' of the norms $|\eta_\alpha|_{0,\omega}$ and $\|\eta_3\|_{1,\omega}$ on the right-hand side of the above inequality, that is, situations where the seminorm

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

becomes a *norm*, which should in addition be *equivalent* to the product norm.

To this end, we begin by establishing (as in dimension three; *cf.* Theorem 3.3) a linearized rigid displacement lemma (Theorem 4.3), which provides in particular one instance of boundary conditions implying that this seminorm becomes a norm; as stated here, this lemma is due to Bernadou and Ciarlet (1976, Theorems 5.1-1 and 5.2-1).

The elegant proof of this lemma given here is based on an idea of Chapelle (1994). It relies on the preliminary observation that a vector field $\eta_i \boldsymbol{a}^i$ on a surface may be 'canonically' extended to a three-dimensional vector field $v_i \boldsymbol{g}^i$, in such a way that all the components $e_{i\|j}(\boldsymbol{v})$ of the associated three-dimensional linearized change of metric tensor have remarkable expressions in terms of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the linearized change of metric and linearized change of curvature tensors of the surface field.

Theorem 4.2: 'Canonical' three-dimensional extension of a surface vector field. Let the assumptions on the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be as in Theorem 4.1 and let

$$\boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}.$$

There exists $\varepsilon_0 > 0$ such that the mapping $\boldsymbol{\Theta} : \overline{\omega} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^3$ defined by

$$\boldsymbol{\Theta}(y, x_3) := \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y) \text{ for all } (y, x_3) \in \overline{\omega} \times [-\varepsilon_0, \varepsilon_0]$$

is a $\mathcal{C}^1$-diffeomorphism. With any vector field $\eta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ with covariant components η_α in $H^1(\omega)$ and $\eta_3 \in H^2(\omega)$, let there be associated the vector field $v_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ defined by

$$v_i(y, x_3) \boldsymbol{g}^i(y, x_3) = \eta_i(y) \boldsymbol{a}^i(y) + x_3 \mathcal{X}_\alpha(y) \boldsymbol{a}^\alpha(y)$$

for all $(y, x_3) \in \overline{\Omega}$, where $\Omega := \omega \times]-\varepsilon_0, \varepsilon_0[$, the vectors $\boldsymbol{g}^i$ form the contravariant basis associated with the mapping $\boldsymbol{\Theta}$ (Section 3), and

$$\mathcal{X}_\alpha := -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma).$$

Then the covariant components v_i of the vector field $v_i \boldsymbol{g}^i$ are in $H^1(\Omega)$ and the covariant components $e_{i\|j}(\boldsymbol{v}) \in L^2(\Omega)$ of the associated linearized change of metric tensor are given by

$$\begin{aligned}
e_{\alpha\|\beta}(\boldsymbol{v}) &= \gamma_{\alpha\beta}(\boldsymbol{\eta}) - x_3 \rho_{\alpha\beta}(\boldsymbol{\eta}) \\
&\quad + \frac{x_3^2}{2} \left\{ b_\alpha^\sigma \rho_{\beta\sigma}(\boldsymbol{\eta}) + b_\beta^\tau \rho_{\alpha\tau}(\boldsymbol{\eta}) - 2 b_\alpha^\sigma b_\beta^\tau \gamma_{\sigma\tau}(\boldsymbol{\eta}) \right\}, \\
e_{i\|3}(\boldsymbol{v}) &= 0.
\end{aligned}$$

Proof. **(i)** *Preliminaries.* The mapping $\boldsymbol{\Theta} : \overline{\omega} \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$ defined above is a $\mathcal{C}^1$-diffeomorphism if $\varepsilon > 0$ is sufficiently small; *cf.* Ciarlet (2000, Theorem 3.1-1). Since

$$\partial_\alpha \boldsymbol{a}_3 = -b_\alpha^\sigma \boldsymbol{a}_\sigma$$

by Weingarten's formula, the vectors of the covariant basis associated with the mapping $\boldsymbol{\Theta} = \boldsymbol{\theta} + x_3 \boldsymbol{a}_3$ are given by

$$\boldsymbol{g}_\alpha = \boldsymbol{a}_\alpha - x_3 b_\alpha^\sigma \boldsymbol{a}_\sigma \text{ and } \boldsymbol{g}_3 = \boldsymbol{a}_3.$$

(ii) Given functions $\eta_\alpha, \mathcal{X}_\alpha \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$, let the vector field $v_i \boldsymbol{g}^i : \overline{\Omega} \to \mathbb{R}^3$ be defined by

$$v_i \boldsymbol{g}^i = \eta_i \boldsymbol{a}^i + x_3 \mathcal{X}_\alpha \boldsymbol{a}^\alpha$$

(in other words, we momentarily ignore the specific forms of the functions $\mathcal{X}_\alpha$ indicated in the theorem). Then the functions v_i are in $H^1(\Omega)$ and the covariant components $e_{i\|j}(\boldsymbol{v})$ of the linearized change of metric tensor associated with the field $v_i \boldsymbol{g}^i$ are given by

$$\begin{aligned}
e_{\alpha\|\beta}(\boldsymbol{v}) \ &= \ \frac{1}{2}\left\{(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3\right\} \\
&+ x_3 \left\{ \frac{1}{2}(\mathcal{X}_{\alpha|\beta} + \mathcal{X}_{\beta|\alpha}) - \frac{1}{2} b_\alpha^\sigma (\eta_{\sigma|\beta} - b_{\beta\sigma}\eta_3) - \frac{1}{2} b_\beta^\tau (\eta_{\tau|\alpha} - b_{\alpha\tau}\eta_3) \right\} \\
&+ \frac{x_3^2}{2}\left\{ -b_\alpha^\sigma \mathcal{X}_{\sigma|\beta} - b_\beta^\tau \mathcal{X}_{\tau|\alpha} \right\}, \\
e_{\alpha\|3}(\boldsymbol{v}) \ &= \ \frac{1}{2}(\mathcal{X}_\alpha + \partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma), \\
e_{3\|3}(\boldsymbol{v}) \ &= \ 0.
\end{aligned}$$

The assumed regularities of the functions η_i and $\mathcal{X}_\alpha$ imply that

$$v_i = (v_j \boldsymbol{g}^j) \cdot \boldsymbol{g}_i = (\eta_i \boldsymbol{a}^i + x_3 \mathcal{X}_\alpha \boldsymbol{a}^\alpha) \cdot \boldsymbol{g}_i \in H^1(\Omega),$$

since $\boldsymbol{g}_i \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^3)$. The stated expressions for the functions $e_{i\|j}(\boldsymbol{v})$ are obtained by simple computations, based on the relations

$$e_{i\|j}(\boldsymbol{v}) = \frac{1}{2}(v_{i\|j} + v_{j\|i}) \text{ and } v_{i\|j} = \left\{\partial_j (v_k \boldsymbol{g}^k)\right\} \cdot \boldsymbol{g}_i$$

(the vectors $\boldsymbol{g}_i$ having been computed in (i)).

(iii) When $\mathcal{X}_\alpha = -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)$, the functions $e_{i\|j}(\boldsymbol{v})$ found in (ii) take the expressions stated in the theorem.

We first note that $\mathcal{X}_\alpha \in H^1(\omega)$ (since $b_\alpha^\sigma \in \mathcal{C}^1(\overline{\omega})$) and that $e_{\alpha\|3}(\boldsymbol{v}) = 0$ when $\mathcal{X}_\alpha = -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma)$. It thus remains to find the explicit forms of the functions $e_{\alpha\|\beta}(\boldsymbol{v})$. Replacing the functions $\mathcal{X}_\alpha$ by their expressions and using the symmetry relations $b^\sigma_\alpha|_\beta = b^\sigma_\beta|_\alpha$, we find that

$$\frac{1}{2}(\mathcal{X}_{\alpha|\beta} + \mathcal{X}_{\beta|\alpha}) - \frac{1}{2}b^\sigma_\alpha(\eta_{\sigma|\beta} - b_{\beta\sigma}\eta_3) - \frac{1}{2}b^\tau_\beta(\eta_{\tau|\alpha} - b_{\alpha\tau}\eta_3)$$
$$= -\eta_{3|\alpha\beta} - \beta^\sigma_\alpha\eta_{\sigma|\beta} - b^\tau_\beta\eta_{\tau|\alpha} - b^\tau_\beta|_\alpha\eta_\tau + b^\sigma_\alpha b_{\sigma\beta}\eta_3,$$

that is, the factor of x_3 in $e_{\alpha\|\beta}(\boldsymbol{v})$ is precisely equal to $-\rho_{\alpha\beta}(\boldsymbol{\eta})$. Finally,

$$\begin{aligned}
-b^\sigma_\alpha&\mathcal{X}_{\sigma|\beta} - b^\tau_\beta\mathcal{X}_{\tau|\alpha} \\
&= b^\sigma_\alpha(\eta_{3|\beta\sigma} + b^\tau_{\sigma|\beta}\eta_\tau + b^\tau_\alpha\eta_{\tau|\beta}) + b^\tau_\beta(\eta_{3|\alpha\tau} + b^\sigma_\tau|_\alpha\eta_\sigma + b^\sigma_\tau\eta_{\sigma|\alpha}) \\
&= b^\sigma_\alpha(\rho_{\beta\sigma}(\boldsymbol{\eta}) - b^\tau_\beta\eta_{\tau|\sigma} + b^\tau_\beta b_{\tau\sigma}\eta_3) + b^\tau_\beta(\rho_{\alpha\tau}(\boldsymbol{\eta}) - b^\sigma_\alpha\eta_{\sigma|\tau} + b^\sigma_\alpha b_{\sigma\tau}\eta_3) \\
&= b^\sigma_\alpha\rho_{\beta\sigma}(\boldsymbol{\eta}) + b^\tau_\beta\rho_{\alpha\tau}(\boldsymbol{\eta}) - 2b^\sigma_\alpha b^\tau_\beta\gamma_{\sigma\tau}(\boldsymbol{\eta}),
\end{aligned}$$

that is, the factor of $x_3^2/2$ in $e_{\alpha\|\beta}(\boldsymbol{v})$ is precisely the combination of functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ stated in the theorem. □

We now establish a linearized rigid displacement lemma on a general surface. 'Linearized' reminds us that only the *linearized* parts of the change of metric and change of curvature tensors are required to vanish. Thanks to Theorem 4.2 this lemma becomes a direct corollary to the 'three-dimensional' linearized rigid displacement lemma (Theorem 3.3), to which it may be profitably compared.

Part (a) of the next theorem is a linearized rigid displacement lemma *without boundary conditions*, while part (b) is a linearized rigid displacement lemma *with boundary conditions*.

Theorem 4.3: Linearized rigid displacement lemma on a general surface. Let the assumptions on the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be as in Theorem 4.1.

(a) Let $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ be such that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega.$$

Then the vector field $\eta_i\boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ is a 'linearized rigid displacement' of the surface $S = \boldsymbol{\theta}(\overline{\omega})$, in the sense that there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$\eta_i(y)\boldsymbol{a}^i(y) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \widehat{\boldsymbol{\theta}}(y) \text{ for all } y \in \overline{\omega}.$$

(b) Let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial\omega$ that satisfies

$$\text{length } \gamma_0 > 0.$$

Then

$$\left.\begin{aligned}
\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega), \\
\eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0, \\
\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega
\end{aligned}\right\} \Rightarrow \boldsymbol{\eta} = \boldsymbol{0} \text{ in } \omega.$$

Proof. Let the set $\Omega = \omega \times]-\varepsilon_0, \varepsilon_0[$ and the field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$ be defined as in Theorem 4.2. By this theorem,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \Rightarrow e_{i\|j}(\boldsymbol{v}) = 0 \text{ in } \Omega,$$

and thus, by Theorem 3.3(a), there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$v_i(y, x_3)\boldsymbol{g}^i(y, x_3) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \{\boldsymbol{\theta}(y) + x_3\boldsymbol{a}_3(y)\} \text{ for all } (y, x_3) \in \overline{\Omega}.$$

Hence

$$\eta_i(y)\boldsymbol{a}^i(y) = v_i(y, x_3)\boldsymbol{g}^i(y, x_3)|_{x_3=0} = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \boldsymbol{\theta}(y) \text{ for all } y \in \overline{\omega},$$

and part (a) is established.

If in addition $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0, then $\mathcal{X}_\alpha = -(\partial_\alpha \eta_3 + b_\alpha^\sigma \eta_\sigma) = 0$ on γ_0, since $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0 implies $\partial_\alpha \eta_3 = 0$ on γ_0; consequently,

$$v_i = (v_j\boldsymbol{g}^j) \cdot \boldsymbol{g}_i = (\eta_j\boldsymbol{a}^j + x_3\mathcal{X}_\alpha\boldsymbol{a}^\alpha) \cdot \boldsymbol{g}_i = 0 \text{ on } \Gamma_0 := \gamma_0 \times [-\varepsilon_0, \varepsilon_0].$$

Since area $\Gamma_0 > 0$, Theorem 3.3(b) implies that $\boldsymbol{v} = \boldsymbol{0}$ in $\overline{\Omega}$, hence that $\boldsymbol{\eta} = \boldsymbol{0}$ on $\overline{\omega}$. □

We are now in a position to prove an inequality that plays a fundamental role in the analysis of linearly elastic shells, in particular in establishing the existence and uniqueness of the solution to the two-dimensional shell equations of W. T. Koiter and of the solution to the two-dimensional equations of a 'flexural' shell, as already observed at the beginning of this section. This inequality is due to Bernadou and Ciarlet (1976); see also Bernadou, Ciarlet and Miara (1994).

Theorem 4.4: Inequality of Korn's type on a general surface. Let the assumptions on the mapping $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ be as in Theorem 4.1, let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial\omega$ that satisfies

$$\text{length } \gamma_0 > 0,$$

and let the space $\mathbf{V}_K(\omega)$ be defined by

$$\mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Then there exists a constant $c = c(\omega, \gamma_0, \boldsymbol{\theta})$ such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}^{1/2} \leq c \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$.

Proof. Let

$$\|\boldsymbol{\eta}\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} := \left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}^{1/2},$$

and let

$$|\boldsymbol{\eta}|_\omega^K := \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}.$$

If the stated inequality is false, there exists a sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ of functions $\boldsymbol{\eta}^k \in \mathbf{V}_K(\omega)$ such that

$$\|\boldsymbol{\eta}^k\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} = 1 \text{ for all } k \text{ and } \lim_{k\to\infty} |\boldsymbol{\eta}^k|_\omega^K = 0.$$

Since the sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ is bounded in $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$, there exists a subsequence $(\boldsymbol{\eta}^l)_{l=1}^\infty$ that converges in $L^2(\omega)\times L^2(\omega)\times H^1(\omega)$ by the Rellich–Kondrašov theorem; furthermore, since $\lim_{l\to\infty} |\boldsymbol{\eta}^l|_\omega^K = 0$, each sequence $(\gamma_{\alpha\beta}(\boldsymbol{\eta}^l))_{l=1}^\infty$ and $(\rho_{\alpha\beta}(\boldsymbol{\eta}^l))_{l=1}^\infty$ also converges in $L^2(\omega)$ (to 0, but this information is not used at this stage). The subsequence $(\boldsymbol{\eta}^l)_{l=1}^\infty$ is thus a Cauchy sequence with respect to the norm

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_{\alpha} |\eta_\alpha|_{0,\omega}^2 + \|\eta_3\|_{1,\omega}^2 + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 + \sum_{\alpha,\beta} |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2},$$

and hence with respect to the norm $\|\cdot\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)}$ by Korn's inequality without boundary conditions (Theorem 4.1).

The space $\mathbf{V}_K(\omega)$ being complete as a closed subspace of $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$, there exists $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$ such that

$$\boldsymbol{\eta}^l \to \boldsymbol{\eta} \text{ in } H^1(\omega)\times H^1(\omega)\times H^2(\omega),$$

and the limit $\boldsymbol{\eta}$ satisfies

$$\begin{aligned} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega} &= \lim_{l\to\infty} |\gamma_{\alpha\beta}(\boldsymbol{\eta}^l)|_{0,\omega} = 0, \\ |\rho_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega} &= \lim_{l\to\infty} |\rho_{\alpha\beta}(\boldsymbol{\eta}^l)|_{0,\omega} = 0. \end{aligned}$$

Hence $\boldsymbol{\eta} = \mathbf{0}$ by Theorem 4.3.

But this contradicts the relations $\|\boldsymbol{\eta}^l\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} = 1$ for all $l \geq 1$, and the proof is complete. □

It has recently been shown by Ciarlet and Mardare (2000, 200x) that the canonical three-dimensional extension of a surface vector field used in Theorem 4.2 can be further put to use, to the extent that it provides a new proof of the inequality of Korn's type on a general surface itself (Theorem 4.4), directly as a corollary to the three-dimensional Korn inequality in curvilinear coordinates (Theorem 3.4).

For another, 'intrinsic', approach to inequalities of Korn's type on surfaces, see Delfour (200x).

5. Inequality of Korn's type on a surface with little regularity

As shown by Blouza and Le Dret (1999), the *regularity* assumptions made in the previous section on the mapping $\boldsymbol{\theta}$ and on the field $\boldsymbol{\eta} = (\eta_i)$, in both the linearized rigid displacement lemma and the inequality of Korn's type (Theorems 4.3 and 4.4), can be substantially weakened.

This improvement relies on the observation that the covariant components of the linearized change of metric and change of curvature tensors, that is,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

and

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) \;=\; & \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 \\ & + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) \\ & + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau,\end{aligned}$$

can be also written as

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\alpha + \partial_\alpha\widetilde{\boldsymbol{\eta}}\cdot\boldsymbol{a}_\beta) =: \widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$$

and

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) = (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}})\cdot\boldsymbol{a}_3 =: \widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}),$$

in terms of the field

$$\widetilde{\boldsymbol{\eta}} := \eta_i\boldsymbol{a}^i.$$

The interest of the new expressions $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ is that they still define *bona fide* distributions under significantly weaker smoothness assumptions than those made so far, that is, $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ and $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$. More specifically, it is easily verified that $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in L^2(\omega)$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in H^{-1}(\omega)$ if $\boldsymbol{\theta} \in W^{2,\infty}(\omega;\mathbb{R}^3)$ and $\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1(\omega)$.

Note that, to avoid any confusion, we intentionally employ the new notation $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$.

Using this observation, Blouza and Le Dret (1999, Theorem 6) first establish the following extension of Theorem 4.3.

Theorem 5.1: Linearized rigid displacement lemma on a surface with little regularity. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in W^{2,\infty}(\omega;\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$.

Given $\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1(\omega)$, let the distributions $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in L^2(\omega)$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in H^{-1}(\omega)$ be defined by

$$\begin{aligned}\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) &:= \frac{1}{2}(\partial_\beta\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta),\\ \widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) &:= (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3.\end{aligned}$$

Let $\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1(\omega)$ be such that

$$\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = 0 \text{ in } \omega.$$

Then $\widetilde{\boldsymbol{\eta}}$ is a 'linearized rigid displacement' of the surface $S = \boldsymbol{\theta}(\overline{\omega})$, in the sense that there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{\eta}}(y) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \boldsymbol{\theta}(y) \text{ for all } y \in \overline{\omega}. \qquad \square$$

Blouza and Le Dret (1999, Lemma 11) then proceed to establish the following variant of Theorem 4.4, which, for convenience, is stated here with boundary conditions corresponding to a shell that is simply supported along its entire boundary, that is, $\widetilde{\boldsymbol{\eta}} = \mathbf{0}$ on γ.

Boundary conditions of *clamping*, as considered in Theorem 4.4, can also be handled via the present approach, provided they are first re-interpreted so as to make sense for vector fields $\widetilde{\boldsymbol{\eta}}$ that only satisfy $\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1(\omega)$ and $\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in L^2(\omega)$; *cf.* Blouza and Le Dret (1999, Section 6).

Theorem 5.2: Inequality of Korn's type on a surface with little regularity. Let the assumptions on the mapping $\boldsymbol{\theta}$ be as in Theorem 5.1 and let the space $\widetilde{\mathbf{V}}^s_K(\omega)$ be defined by

$$\widetilde{\mathbf{V}}^s_K(\omega) = \{\widetilde{\boldsymbol{\eta}} \in \mathbf{H}^1_0(\omega) : \partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in L^2(\omega)\}.$$

Then there exists a constant c such that

$$\begin{aligned}&\left\{\|\widetilde{\boldsymbol{\eta}}\|^2_{1,\omega} + \sum_{\alpha,\beta}|\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3|^2_{0,\omega}\right\}^{1/2}\\ &\qquad \le c\left\{\sum_{\alpha,\beta}|\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})|^2_{0,\omega} + \sum_{\alpha,\beta}|\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})|^2_{0,\omega}\right\}^{1/2} \text{ for all } \widetilde{\boldsymbol{\eta}} \in \widetilde{\mathbf{V}}^s_K(\omega),\end{aligned}$$

where the distributions $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ are defined as in Theorem 5.1 (note that $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) \in L^2(\omega)$ if $\widetilde{\boldsymbol{\eta}} \in \widetilde{\mathbf{V}}^s_K(\omega)$). $\square$

This theorem establishes as a corollary the existence and uniqueness of the solution to the two-dimensional Koiter equations for a simply supported shell whose middle surface has little regularity, once these equations are re-written in terms of the expressions $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$; *cf.* Section 11.3.

6. Inequality of Korn's type on an elliptic surface

The two-dimensional equations of a linearly elastic 'membrane' shell take the following form. The unknowns are the covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the displacement $\zeta_i^\varepsilon \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ of the shell, and $\boldsymbol{\zeta}^\varepsilon := (\zeta_i^\varepsilon)$ satisfies

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$
$$\int_\omega \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y = \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y,$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$, where $2\varepsilon > 0$ is the thickness of the shell,

$$a^{\alpha\beta\sigma\tau,\varepsilon} := \frac{4\lambda^\varepsilon \mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^\varepsilon (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

denote (as in Section 4) the contravariant components of the two-dimensional elasticity tensor of the shell,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta} \eta_3$$

denote (again as in Section 4) the covariant components of the linearized change of metric tensor of S, and the given functions $p^{i,\varepsilon} \in L^2(\omega)$ account for the applied forces.

These equations will be further studied in Section 8, where it will be shown in particular that they can be fully justified through an asymptotic analysis of the three-dimensional solutions.

As already noted in Section 4, there exist constants c_e and a_0 such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \le c_e a^{\alpha\beta\sigma\tau,\varepsilon}(y) t_{\sigma\tau} t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ and such that $a(y) \ge a_0 > 0$ for all $y \in \overline{\omega}$. Establishing the existence and uniqueness of a solution to the above variational problem by the Lax–Milgram lemma thus amounts to proving the existence of a constant c_M such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + |\eta_3|_{0,\omega}^2 \right\}^{1/2} \le c_M \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$.

The objective of this section, based on Ciarlet and Lods (1996a) and Ciarlet and Sanchez-Palencia (1996), is to find sufficient conditions, essentially bearing on the 'geometry' of the surface S, guaranteeing that such an inequality of Korn's type holds. It is also worth noticing that the justification alluded to above of these two-dimensional 'membrane' shell equations from three-dimensional elasticity is performed under precisely the *same* assumptions on the geometry of S, as we shall see in Section 8.

We follow the usual pattern, that is, we begin by proving an inequality of Korn's type *without boundary condition*, which remarkably holds for 'arbitrary' geometries, although it only involves the linearized change of metric tensor (compare with Theorem 4.1).

Theorem 6.1: Second inequality of Korn's type without boundary conditions on a general surface. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$. Given $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$, let

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \left\{ \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3 \right\} \in L^2(\omega)$$

denote the covariant components of the linearized change of metric tensor associated with the displacement field $\eta_i \boldsymbol{a}^i$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$. Then there exists a constant $c_0 = c_0(\omega, \boldsymbol{\theta})$ such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + |\eta_3|^2_{0,\omega} \right\}^{1/2} \leq c_0 \left\{ \sum_i |\eta_i|^2_{0,\omega} + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$.

Proof. The proof is analogous to that of Theorem 4.1 and, for this reason, is only sketched. It relies on the following steps. First, the space

$$\mathbf{W}_M(\omega) := \left\{ \boldsymbol{\eta} = (\eta_i) \in \mathbf{L}^2(\omega) : \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega) \right\}$$

becomes a Hilbert space when it is equipped with the norm $\|\cdot\|^M_\omega$ defined by

$$\|\boldsymbol{\eta}\|^M_\omega := \left\{ \sum_i |\eta_i|^2_{0,\omega} + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} \right\}^{1/2}.$$

Next, the two spaces $\mathbf{W}_M(\omega)$ and $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ coincide, thanks again to the identities

$$\partial_{\alpha\beta} \eta_\sigma = \partial_\alpha e_{\beta\sigma}(\boldsymbol{\eta}) + \partial_\beta e_{\alpha\sigma}(\boldsymbol{\eta}) - \partial_\sigma e_{\alpha\beta}(\boldsymbol{\eta})$$

and to the Lemma of J. L. Lions (Theorem 3.1).

Finally, the closed graph theorem shows that the identity mapping from the space $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ equipped with the product norm

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + |\eta_3|^2_{0,\omega} \right\}^{1/2}$$

onto the space $\mathbf{W}_M(\omega)$ equipped with the norm $\|\cdot\|_\omega^M$ has a continuous inverse. Hence the stated inequality holds. $\square$

The next step consists in identifying sufficient conditions allowing the 'elimination' of the norms $|\eta_i|_{0,\omega}$ on the right-hand side of the above inequality of Korn's type. Whether it be for the three-dimensional Korn inequality in curvilinear coordinates (Theorem 3.4) or for the inequality of Korn's type on a general surface (Theorem 4.4), the corresponding eliminations simply resulted from imposing *ad hoc* boundary conditions on the displacement fields, in such a way that a linearized rigid displacement lemma with boundary conditions holds (see Theorems 3.3(b) and 4.3(b)).

In other words, we are facing the problem of finding boundary conditions such that the seminorm

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

becomes a *norm* for the displacement fields $\eta_i \boldsymbol{a}^i$ that satisfy them. Since η_3 is only in $L^2(\omega)$ and η_α is in $H^1(\omega)$, the only possibility consists in trying boundary conditions of the form

$$\eta_\alpha = 0 \text{ on } \gamma_0 \subset \gamma, \text{ with area } \gamma_0 > 0.$$

It then turns out that such a linearized rigid displacement lemma does hold, but only for *special geometries of the surface* S and *special subsets* $\boldsymbol{\theta}(\gamma_0)$ of *the boundary of* S. In this direction, we refer to Sanchez-Palencia (1993), Sanchez-Hubert and Sanchez-Palencia (1997), Lods and Mardare (1998*a*), Mardare (1998*c*), and Şlicaru (1998), who have identified various situations of interest where this lemma holds.

But even though such a linearized rigid displacement lemma *often* holds, it *very seldom* implies that the norm

$$\boldsymbol{\eta} \to \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

is equivalent to the norm

$$\boldsymbol{\eta} \to \left\{ \sum_{\alpha} \|\eta_\alpha\|_{1,\omega}^2 + |\eta_3|_{0,\omega}^2 \right\}^{1/2}.$$

More precisely, we shall prove (in Theorem 6.3) that, under *ad hoc* regularity assumptions on the mapping $\boldsymbol{\theta}$ and on the boundary γ, these two norms are equivalent if $\gamma = \gamma_0$ and the surface S is *elliptic* according to the definition given below. Conversely, Şlicaru (1998) has shown the remarkable result that, even under the 'minimal' regularity assumptions '$\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$

and γ Lipschitz-continuous', the same sufficient conditions are also necessary for the equivalence of the norms, which thus very seldom occurs indeed!

We now prove the stated 'linearized rigid displacement lemma', directly under the assumptions ($\gamma_0 = \gamma$ and S elliptic) that will eventually lead to the equivalence of norms. We begin with a definition.

Let a surface $S = \boldsymbol{\theta}(\overline{\omega})$ be given, where $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ is an injective mapping such that the two vectors $\boldsymbol{a}_\alpha$ are linearly independent at all points of $\overline{\omega}$. Then S is *elliptic* if the symmetric matrix $(b_{\alpha\beta}(y))$ formed by the covariant components of the curvature tensor of S is positive, *or* negative, definite at all points $y \in \overline{\omega}$, or equivalently, if there exists a constant c such that

$$c > 0 \text{ and } |b_{\alpha\beta}(y)\xi^\alpha\xi^\beta| \geq c \sum_\alpha |\xi^\alpha|^2,$$

for all $y \in \overline{\omega}$ and all $(\xi^\alpha) \in \mathbb{R}^2$. Geometrically, this means that the *Gaussian curvature* of the surface S is everywhere > 0, or equivalently, that the two principal radii of curvature are of the same sign at each point of S (for details about these classical notions, see, *e.g.*, Ciarlet (2000, Section 2.2)). A portion of an *ellipsoid* provides an instance of elliptic surface.

In the next proof of the theorem, analytic functions of two real variables in an open subset of $\mathbb{R}^2$ are considered; we refer to Dieudonné (1968) for a particularly elegant treatment of analytic functions of any finite number of real or complex variables.

Theorem 6.2: Linearized rigid displacement lemma on an elliptic surface. Let there be given a domain ω in $\mathbb{R}^2$ and an injective mapping $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega}; \mathbb{R}^3)$ such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$ and such that the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is elliptic. Then

$$\left.\begin{aligned} \boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega), \\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \end{aligned}\right\} \Rightarrow \boldsymbol{\eta} = \mathbf{0} \text{ in } \omega.$$

Proof. We give the proof under the additional assumptions that the boundary γ is of class $\mathcal{C}^3$ and that the components of the mapping $\boldsymbol{\theta}$ are restrictions to $\overline{\omega}$ of analytic functions in an open set $\omega' \subset \mathbb{R}^2$ containing $\overline{\omega}$. We refer to Lods and Mardare (1998*a*) for a proof (then more 'technical') under the more general assumptions stated in the theorem. An earlier version of this lemma is due to Vekua (1962), who proved it under the assumptions that γ is of class $\mathcal{C}^3$ and $\boldsymbol{\theta} \in W^{3,p}(\omega; \mathbb{R}^3)$ for some $p > 1$, using the theory of 'generalized analytic functions'.

(i) We first note that establishing this implication is equivalent to proving a *uniqueness theorem*, that is, $\boldsymbol{\eta} = (\eta_i) = \mathbf{0}$ is the only solution in the space $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ of the linear system formed by the three partial differential equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω together with the two boundary conditions (understood in the sense of traces) $\eta_\alpha = 0$ on γ, or, 'in full',

$$\begin{aligned} \partial_1\eta_1 - \Gamma^\sigma_{11}\eta_\sigma - b_{11}\eta_3 &= 0 \text{ in } \omega, \\ \frac{1}{2}\partial_2\eta_1 + \frac{1}{2}\partial_1\eta_2 - \Gamma^\sigma_{12}\eta_\sigma - b_{12}\eta_3 &= 0 \text{ in } \omega, \\ \partial_2\eta_2 - \Gamma^\sigma_{22}\eta_\sigma - b_{22}\eta_3 &= 0 \text{ in } \omega, \\ \eta_1 &= 0 \text{ on } \gamma, \\ \eta_2 &= 0 \text{ on } \gamma. \end{aligned}$$

(ii) Any solution $\boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$ of the system

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \text{ and } \eta_\alpha = 0 \text{ on } \gamma$$

is in the space $\mathcal{C}^1(\overline{\omega}) \times \mathcal{C}^1(\overline{\omega}) \times \mathcal{C}^0(\overline{\omega})$.

This regularity result relies on a crucial observation made by Geymonat and Sanchez-Palencia (1991). The partial differential equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω constitute a first-order system that is 'uniformly elliptic' in the sense of Agmon, Douglis and Nirenberg (1964). This means that there exists a constant $A > 0$ such that (here and subsequently, we use the notation of Agmon, Douglis and Nirenberg (1964))

$$A^{-1}\sum_\alpha |\xi_\alpha|^2 \leq |L(y,\boldsymbol{\xi})| \leq A\sum_\alpha |\xi_\alpha|^2,$$

for all $y \in \overline{\omega}$ and $\boldsymbol{\xi} = (\xi_\alpha) \in \mathbb{R}^2$, where

$$L(y,\boldsymbol{\xi}) := \det\begin{pmatrix} \xi_1 & 0 & -b_{11}(y) \\ \frac{1}{2}\xi_2 & \frac{1}{2}\xi_1 & -b_{12}(y) \\ 0 & \xi_2 & -b_{22}(y) \end{pmatrix}.$$

The way the above matrix of order three is constructed from the equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ should be clear; suffice it to specify that only the coefficients of the partial derivatives of the highest order for each unknown (one for η_α and zero for η_3) are taken into account. The uniform ellipticity of the system formed by the partial differential equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω thus holds, since

$$L(y,\boldsymbol{\xi}) = -\frac{1}{2}(\xi_2 - \xi_1)\begin{pmatrix} b_{11}(y) & b_{12}(y) \\ b_{21}(y) & b_{22}(y) \end{pmatrix}\begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix}$$

in the present case, and since the symmetric matrix $(b_{\alpha\beta}(y))$ is either positive, or negative, definite at all points $y \in \overline{\omega}$ by the assumed *ellipticity* of the surface S.

In addition, the 'supplementary condition on L' (which needs to be verified only in two dimensions, as here) is also satisfied. The degree m of the polynomial L with respect to ξ_1 and ξ_2 being two, the polynomial

$$\tau \in \mathbb{C} \to L(y, \boldsymbol{\xi} + \tau\boldsymbol{\eta}) \in \mathbb{C}$$

has exactly $\frac{m}{2} = 1$ root τ^+ with $\operatorname{Im}\tau^+ > 0$, for all $y \in \overline{\omega}$ and all linearly independent vectors $\boldsymbol{\xi} = (\xi_\alpha)$ and $\boldsymbol{\eta} = (\eta_\alpha)$ in $\mathbb{R}^2$.

Finally, when $\frac{m}{2}$, *i.e.*, *one* of the two boundary conditions $\eta_\alpha = 0$ on γ is appended to the equations $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω, the 'complementary boundary condition' is also satisfied. Thus the polynomial $\tau \in \mathbb{C} \to (\tau - \tau^+)$ divides the polynomials $\tau \to c(\xi_1 + \tau\eta_1)$ and $\tau \to c(\xi_2 + \tau\eta_2)$ only if the constant c vanishes.

It then follows from Agmon, Douglis and Nirenberg (1964, Theorem 10.5) that, if γ is of class $\mathcal{C}^3$ and the coefficients of the uniformly elliptic system $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ are in $\mathcal{C}^2(\overline{\omega})$, any solution $\boldsymbol{\eta} \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ of $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω together with, for instance, $\eta_1 = 0$ on γ is in the space $H^3(\omega) \times H^3(\omega) \times H^2(\omega)$. The assertion then follows from the continuous embeddings $H^m(\omega) \hookrightarrow \mathcal{C}^{m-2}(\overline{\omega}), m = 2, 3$.

(iii) 'Local' uniqueness of the solution of the system

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \text{ and } \eta_\alpha = 0 \text{ on } \gamma.$$

The assumed ellipticity of the surface S shows that there exists a constant $c > 0$ such that $|b_{11}(y)| \geq c$ for all $y \in \overline{\omega}$. Hence the unknown η_3 may be eliminated, for instance by means of the equation $\gamma_{11}(\boldsymbol{\eta}) = 0$. This elimination shows that

$$\eta_3 = \frac{1}{b_{11}}(\partial_1\eta_1 - \Gamma_{11}^\sigma\eta_\sigma)$$

and that η_1 and η_2 are solutions of the *reduced system*

$$\begin{aligned}
-2\frac{b_{12}}{b_{11}}\partial_1\eta_1 + \partial_2\eta_1 + \partial_1\eta_2 - 2\left(\Gamma_{12}^\sigma - \frac{b_{12}}{b_{11}}\Gamma_{11}^\sigma\right)\eta_\sigma &= 0 \text{ in } \omega,\\
-\frac{b_{22}}{b_{11}}\partial_1\eta_1 + \partial_2\eta_2 - \left(\Gamma_{22}^\sigma - \frac{b_{22}}{b_{11}}\Gamma_{11}^\sigma\right)\eta_\sigma &= 0 \text{ in } \omega,\\
\eta_1 &= 0 \text{ on } \gamma,\\
\eta_2 &= 0 \text{ on } \gamma.
\end{aligned}$$

Since the coefficients of this reduced system are analytic in ω', since the boundary γ is of class $\mathcal{C}^3$ and is not a characteristic curve for this system, as is easily verified by again using the assumed *ellipticity* of the surface S, Holmgren's Uniqueness Theorem (see, *e.g.*, Courant and Hilbert (1962, p. 238) or Bers, John and Schechter (1964, p. 47)) shows that 'locally', *i.e.*, in a sufficiently small neighbourhood $\widetilde{\omega} \subset \omega'$ of any point $\widetilde{y}$ of γ,

$\eta_1 = \eta_2 = 0$ is the unique solution in $\mathcal{C}^1(\widetilde{\omega})$. Recalling that any solution $\boldsymbol{\eta} = (\eta_i)$ of the 'full' system is such that $\eta_\alpha \in \mathcal{C}^1(\overline{\omega})$ by (ii), we have thus reached the following conclusion. Given any point $\widetilde{y} \in \gamma$, there exists a neighbourhood $\widetilde{\omega} \subset \omega$ of $\widetilde{y}$ such that $\boldsymbol{\eta} = \mathbf{0}$ is the only solution $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ in $\widetilde{\omega} \cap \overline{\omega}$ of the 'full' system

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \quad \text{ and } \quad \eta_\alpha = 0 \text{ on } \gamma.$$

(iv) 'Global' uniqueness of the solution of the system

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega \quad \text{ and } \quad \eta_\alpha = 0 \text{ on } \gamma.$$

By a theorem of Morrey and Nirenberg (1957), any solution of a uniformly elliptic system whose coefficients are analytic in ω is analytic in ω. Since $\boldsymbol{\eta} = \mathbf{0}$ is an analytic solution, the Analytic Continuation Theorem for analytic functions of several variables (see, *e.g.*, Dieudonné (1968, Theorem 9.4.2)) thus shows that $\boldsymbol{\eta} = \mathbf{0}$ is the only solution. □

We are now in a position to prove the main result of this section, due to Ciarlet and Lods (1996*a*) and Ciarlet and Sanchez-Palencia (1996), who provided two different proofs. Special mention must also be made of the early existence and uniqueness theorem of Destuynder (1985, Theorems 6.1 and 6.5), obtained under the additional assumptions that the elliptic surface S can be covered by a single system of lines of curvature and that the $\mathcal{C}^0(\overline{\omega})$-norms of the Christoffel symbols of S are sufficiently small.

It is indeed remarkable that, if the surface S is elliptic and the tangential components of the admissible displacement fields of S vanish over the entire boundary of S, the L^2-norm of the linearized change of metric tensor alone is 'already' equivalent to the $(H^1 \times H^1 \times L^2)$-norm of these fields (compare with Theorem 4.4; note, however, that the H^2-norm of the normal components that appears there in the inequality of Korn's type on a general surface is now replaced by the L^2-norm).

Theorem 6.3: Inequality of Korn's type on an elliptic surface. Let the assumptions be as in Theorem 6.2. Then there exists a constant $c_M = c_M(\omega, \boldsymbol{\theta})$ such that

$$\left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + |\eta_3|_{0,\omega}^2 \right\}^{1/2} \le c_M \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$.

Proof. **(i)** By the second inequality of Korn's type without boundary conditions on a general surface (Theorem 6.1), there exists a constant c_0 such that

$$\|\boldsymbol{\eta}\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)} := \Big\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + |\eta_3|^2_{0,\omega}\Big\}^{1/2}$$
$$\leq c_0\Big\{\sum_i |\eta_i|^2_{0,\omega} + \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\Big\}^{1/2}$$

for all $\boldsymbol{\eta} \in \mathbf{V}_M(\omega)$, since $\mathbf{V}_M(\omega) \subset H^1(\omega)\times H^1(\omega)\times L^2(\omega)$. Hence it suffices to show that there exists a constant c such that

$$\Big\{\sum_i |\eta_i|^2_{0,\omega}\Big\}^{1/2} \leq c\Big\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\Big\}^{1/2} \quad \text{for all } \boldsymbol{\eta} \in \mathbf{V}_M(\omega).$$

(ii) If the last inequality is false, there exists a sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ of functions $\boldsymbol{\eta}^k = (\eta_i^k) \in \mathbf{V}_M(\omega)$ such that

$$\Big\{\sum_i |\eta_i^k|^2_{0,\omega}\Big\}^{1/2} = 1 \text{ for all } k \quad\text{and}\quad \lim_{k\to\infty}\Big\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta}^k)|^2_{0,\omega}\Big\}^{1/2} = 0.$$

In particular, then, the sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ is bounded with respect to the norm $\|\cdot\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}$, thanks again to the second inequality of Korn's type of Theorem 6.1. Since any bounded sequence in a Hilbert space contains a weakly convergent sequence, there exists a subsequence $(\boldsymbol{\eta}^l)_{l=1}^\infty$ and an element $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$ such that

$$\eta_\alpha^l \rightharpoonup \eta_\alpha \text{ in } H^1(\omega) \text{ and } \eta_\alpha^l \to \eta_\alpha \text{ in } L^2(\omega),$$
$$\eta_3^l \rightharpoonup \eta_3 \text{ in } L^2(\omega),$$

where $\rightharpoonup$ and $\to$ denote weak and strong convergence (the compact embedding $H^1(\omega) \Subset L^2(\omega)$ is also used here).

(iii) Naturally, the difficulty rests with the subsequence $(\eta_3^l)_{l=1}^\infty$, which converges only *weakly* in $L^2(\omega)$. Our recourse for showing that it in fact *strongly* converges in $L^2(\omega)$ will be (*cf.* (iv)) the assumed *ellipticity of the surface* S; but first, we prove that $\boldsymbol{\eta} = (\eta_i) = \mathbf{0}$. To this end, we simply note that

$$\eta_\alpha^l \rightharpoonup \eta_\alpha \text{ in } H^1(\omega) \text{ and } \eta_3^l \rightharpoonup \eta_3 \text{ in } L^2(\omega) \Rightarrow \gamma_{\alpha\beta}(\boldsymbol{\eta}^l) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{\eta}) \text{ in } L^2(\omega),$$

on the one hand; since

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}^l) \to 0 \text{ in } L^2(\omega),$$

on the other, we conclude that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$. Hence $\boldsymbol{\eta} = \mathbf{0}$ by Theorem 6.2.

(iv) We next show that $\eta_3^l \to 0$ in $L^2(\omega)$. The strong convergence

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}^l) \to 0 \text{ in } L^2(\omega) \text{ and } \eta_\alpha^l \to 0 \text{ in } L^2(\omega)$$

combined with the definition of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ implies the following strong convergence:

$$\begin{aligned}
\partial_1\eta_1^l - b_{11}\eta_3^l &= \{\gamma_{11}(\boldsymbol{\eta}^l) + \Gamma_{11}^\sigma\eta_\sigma^l\} \quad \to 0 \text{ in } L^2(\omega),\\
\partial_2\eta_1^l + \partial_1\eta_2^l - 2b_{12}\eta_3^l &= \{2\gamma_{12}(\boldsymbol{\eta}^l) + 2\Gamma_{12}^\sigma\eta_\sigma^l\} \to 0 \text{ in } L^2(\omega),\\
\partial_2\eta_2^l - b_{22}\eta_3^l &= \{\gamma_{22}(\boldsymbol{\eta}^l) + \Gamma_{22}^\sigma\eta_\sigma^l\} \quad \to 0 \text{ in } L^2(\omega).
\end{aligned}$$

As the function $b_{11} \in \mathcal{C}^0(\overline{\omega})$ does not vanish in $\overline{\omega}$ by the assumed *ellipticity* of the surface S, we can eliminate η_3^l between the first and second, and between the first and third, relations; this elimination yields

$$\begin{aligned}
\left\{\partial_2\eta_1^l + \partial_1\eta_2^l - 2\frac{b_{12}}{b_{11}}\partial_1\eta_1^l\right\} &\to 0 \text{ in } L^2(\omega),\\
\left\{\partial_2\eta_2^l - \frac{b_{22}}{b_{11}}\partial_1\eta_1^l\right\} &\to 0 \text{ in } L^2(\omega).
\end{aligned}$$

Multiplying the first relation by $\partial_2\eta_1^l$ and the second by $\partial_1\eta_1^l$, then integrating over ω, we get

$$\begin{aligned}
\int_\omega \left\{(\partial_2\eta_1^l)^2 + \partial_2\eta_1^l\partial_1\eta_2^l - 2\frac{b_{12}}{b_{11}}\partial_1\eta_1^l\partial_2\eta_1^l\right\} \mathrm{d}y &\to 0,\\
\int_\omega \left\{\partial_1\eta_1^l\partial_2\eta_2^l - \frac{b_{22}}{b_{11}}(\partial_1\eta_1^l)^2\right\} \mathrm{d}y &\to 0,
\end{aligned}$$

since each sequence $(\partial_\alpha\eta_1^l)_{l=1}^\infty$ is bounded in $L^2(\omega)$ (each sequence even weakly converges to 0 in $L^2(\omega)$). Subtracting the last two relations and using the relation $\int_\omega \partial_2\eta_1^l\partial_1\eta_2^l\,\mathrm{d}y = \int_\omega \partial_1\eta_1^l\partial_2\eta_2^l\,\mathrm{d}y$, we thus obtain

$$\int_\omega \left\{\left(\partial_2\eta_1^l - \frac{b_{12}}{b_{11}}\partial_1\eta_1^l\right)^2 + \frac{1}{(b_{11})^2}\big(b_{11}b_{22} - (b_{12})^2\big)\big(\partial_1\eta_1^l\big)^2\right\} \mathrm{d}y \to 0,$$

and consequently

$$\partial_1\eta_1^l \to 0 \text{ in } L^2(\omega),$$

since $b_{11}b_{22} - (b_{12})^2 = \det(b_{\alpha\beta}) \in \mathcal{C}^0(\overline{\omega})$ does not vanish in $\overline{\omega}$ by the assumed ellipticity of S. Hence

$$\eta_3^l = \left\{\frac{1}{b_{11}}\partial_1\eta_1^l - \frac{1}{b_{11}}\big(\partial_1\eta_1^l - b_{11}\eta_3^l\big)\right\} \to 0 \text{ in } L^2(\omega).$$

(v) The relations $\eta_i^l \to 0$ in $L^2(\omega)$ established in parts (ii)–(iv) thus contradict the relations $\{\sum_i |\eta_i^l|_{0,\omega}^2\}^{1/2} = 1$ for all l, and the proof is complete. $\square$

When the surface S is elliptic, the covariant components η_α of the displacement field vanish over the entire boundary γ, and the assumptions on

ω and $\boldsymbol{\theta}$ are as in Theorem 6.2, the two-dimensional equations of a 'membrane' shell (described at the beginning of this section) thus have exactly one solution.

7. Preliminaries to the asymptotic analysis of linearly elastic shells

The purpose of this section is to gather the fundamental preliminaries needed in Sections 8 to 10 for carrying out the asymptotic analysis of all kinds of linearly elastic shells. After *ad hoc* 'scalings' of the unknowns (the covariant components of the three-dimensional displacement field) and *ad hoc* 'asymptotic' assumptions on the data (the Lamé constants and applied force densities) have been made, the problem of a linearly elastic clamped shell with thickness $2\varepsilon > 0$ is transformed into a *scaled problem*, defined over a domain that is *independent of* ε.

Recall that ε is *not* subjected to the rule governing Greek exponents.

7.1. The three-dimensional equations

As in Section 4, let ω be a domain in $\mathbb{R}^2$ with boundary γ, let $y = (y_\alpha)$ denote a generic point in the set $\overline{\omega}$, and let $\partial_\alpha := \partial/\partial y_\alpha$. Let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an *injective* mapping such that the two vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ are *linearly independent at all points* $y \in \overline{\omega}$. These two vectors form the *covariant basis* of the tangent plane to the surface $S := \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$ and the two vectors $\boldsymbol{a}^\alpha(y)$ of the tangent plane at $\boldsymbol{\theta}(y)$ defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$ form its *contravariant basis*. Also, let

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}.$$

Then $|\boldsymbol{a}_3(y)| = 1$, the vector $\boldsymbol{a}_3(y)$ is normal to S at the point $\boldsymbol{\theta}(y)$, and the three vectors $\boldsymbol{a}^i(y)$ form the *contravariant basis at* $\boldsymbol{\theta}(y)$. Recall that (y_1, y_2) constitutes a system of curvilinear coordinates for describing the surface S.

Let γ_0 denote a $d\gamma$-measurable subset of the boundary γ of ω satisfying

$$\text{length } \gamma_0 > 0.$$

For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[\text{ and } \Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon].$$

Let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set $\overline{\Omega}^\varepsilon$ and let $\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon$; hence $x_\alpha^\varepsilon = y_\alpha$ and $\partial_\alpha^\varepsilon = \partial_\alpha$.

Consider an *elastic shell* with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and (constant) thickness $2\varepsilon > 0$, that is, an elastic body whose reference configuration consists of all points within a distance $\leq \varepsilon$ from S. In other words, the reference configuration of the shell is the image $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon) \subset \mathbb{R}^3$ of the set $\overline{\Omega}^\varepsilon \subset \mathbb{R}^3$ through the mapping $\boldsymbol{\Theta} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ given by

$$\boldsymbol{\Theta}(x^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}_3(y) \text{ for all } x^\varepsilon = (y, x_3^\varepsilon) = (y_1, y_2, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon.$$

It can then be shown (Ciarlet 2000, Theorem 3.1-1) that the mapping $\boldsymbol{\Theta} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ is injective for sufficiently small $\varepsilon > 0$. In other words, if $\varepsilon > 0$ is sufficiently small, $(y_1, y_2, x_3^\varepsilon)$ constitutes a *bona fide* system of curvilinear coordinates (Section 3) for describing the reference configuration $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$ of the shell and the physical problem is meaningful since the set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$ 'does not interpenetrate itself'. These curvilinear coordinates are called the *'natural' curvilinear coordinates of the shell* and the curvilinear coordinate $x_3^\varepsilon \in [-\varepsilon, \varepsilon]$ is called the *transverse variable.* We shall also use the notation x_i^ε for the 'natural' curvilinear coordinates of the shell, that is, we shall let $x_\alpha^\varepsilon = y_\alpha$, so that a generic point in the set $\overline{\Omega}^\varepsilon$ may be written as $x^\varepsilon = (x_i^\varepsilon)$.

It can likewise be shown (see Ciarlet (2000, Theorem 3.1-1)) that, again for sufficiently small $\varepsilon > 0$, the three vectors $\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(x^\varepsilon)$ form the *covariant basis* at each point $\boldsymbol{\Theta}(x^\varepsilon), x^\varepsilon \in \overline{\Omega}^\varepsilon$, of the reference configuration, while the three vectors $\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ defined by $\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_j^\varepsilon(x^\varepsilon) = \delta_j^i$ form the *contravariant basis* at $\boldsymbol{\Theta}(x^\varepsilon)$.

As in Section 3, we define the covariant and contravariant components g_{ij}^ε and $g^{ij,\varepsilon}$ of the *metric tensor* of the set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$ and the *Christoffel symbols* $\Gamma_{ij}^{p,\varepsilon}$ by letting

$$g_{ij}^\varepsilon := \boldsymbol{g}_i^\varepsilon \cdot \boldsymbol{g}_j^\varepsilon, \quad g^{ij,\varepsilon} := \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}, \text{ and } \Gamma_{ij}^{p,\varepsilon} := \boldsymbol{g}^{p,\varepsilon} \cdot \partial_i^\varepsilon \boldsymbol{g}_j^\varepsilon$$

(we omit the explicit dependence on x^ε).

The *volume element* in the set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$ is $\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$, where $g^\varepsilon := \det(g_{ij}^\varepsilon)$.

We assume that the material constituting the shell is homogeneous and isotropic and that $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$ is a natural state, so that the material is characterized by its two Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$ (Ciarlet 1988, Section 6.2). The unknown of the problem is the vector field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$, where the three functions $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}$ are the covariant components (with respect to the contravariant bases $\{\boldsymbol{g}^{i,\varepsilon}\}$) of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ experienced by the shell under the influence of *applied forces*. This means that $u_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ is the displacement of the point $\boldsymbol{\Theta}(x^\varepsilon)$; see Figure 7.1.

Finally, we assume that the shell is subjected to a boundary condition of place $\boldsymbol{u}^\varepsilon = \boldsymbol{0}$ on Γ_0^ε, that is, that the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ vanishes along the portion $\boldsymbol{\Theta}(\Gamma_0^\varepsilon)$ of its *lateral face* $\boldsymbol{\Theta}(\gamma \times [-\varepsilon, \varepsilon])$.

In linearized elasticity, the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ then satisfies the following three-dimensional variational problem $\mathcal{P}(\Omega^\varepsilon)$ for a linearly elastic shell in curvilinear coordinates, that is, stated in terms of the 'natural' curvilinear

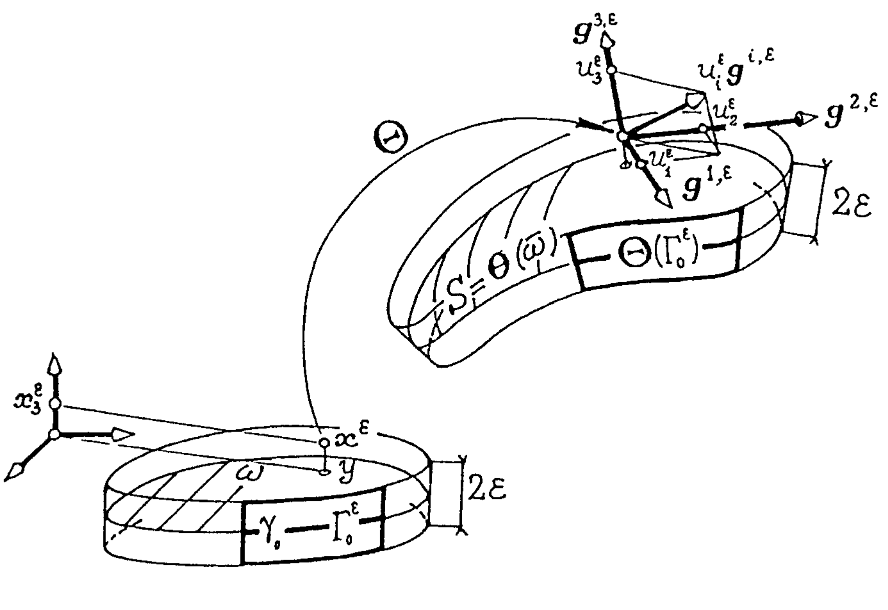

Fig. 7.1. A three-dimensional shell problem. Let $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$. The set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$, where $\boldsymbol{\Theta}(y, x_3^\varepsilon) = \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}_3(y)$ for all $x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon$, is the reference configuration of a shell, with thickness 2ε and middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. The unknowns of the problem are the three covariant components $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}$ of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ of the points of $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$. This means that, for each $x^\varepsilon \in \overline{\Omega}^\varepsilon$, $u_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)$ is the displacement of the point $\boldsymbol{\Theta}(x^\varepsilon) \in \boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$

coordinates x_i^ε of the shell,

$$\boldsymbol{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon) : \boldsymbol{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\},$$
$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) \sqrt{g^\varepsilon}\, \mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\, \mathrm{d}x^\varepsilon$$

for all $\boldsymbol{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$, where

$$A^{ijkl,\varepsilon} := \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon})$$

designate the contravariant components of the *three-dimensional elasticity tensor* of the shell, and

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}\left(\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon\right) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon$$

are the *linearized strains in curvilinear coordinates* associated with an arbitrary displacement field $v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$; $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ denote the contravariant components of the *applied body force density*, applied to the interior $\mathbf{\Theta}(\Omega^\varepsilon)$ of the shell, and $d\Gamma^\varepsilon$ designates the area element along $\partial\Omega^\varepsilon$. See Ciarlet (2000, Chapters 1 and 3) for details.

Surface forces over the 'upper' and 'lower' faces $\mathbf{\Theta}(\omega \times \{\varepsilon\})$ and $\mathbf{\Theta}(\omega \times \{-\varepsilon\})$ of the shell could as well be considered without much further ado, other than 'technical': their consideration simply results in extra terms on the right-hand sides of the two-dimensional equations that will eventually be obtained (see Ciarlet (2000) for details). By contrast, we *assume* that there are *no* surface forces applied to the portion $\mathbf{\Theta}((\gamma - \gamma_0) \times [-\varepsilon, \varepsilon])$ of the lateral face of the shell, as their consideration would substantially modify the subsequent analyses.

The above three-dimensional equations of a linearly elastic shell have exactly one solution. This existence and uniqueness result relies on the three-dimensional Korn inequality in curvilinear coordinates (Theorem 3.4), combined with the uniform positive definiteness of the three-dimensional elasticity tensor, already mentioned in Section 4.

Our basic objective consists in showing that, if $\varepsilon > 0$ is small enough and the data are of appropriate orders with respect to ε, the above three-dimensional problems are 'asymptotically equivalent' to a *two-dimensional problem* posed over the middle surface of the shell. This means that the new unknown should be $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$, where ζ_i^ε are the covariant components (*i.e.*, over the covariant bases $\{\boldsymbol{a}^i\}$) of the displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the points of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. In other words, $\zeta_i^\varepsilon(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\theta}(y) \in S$; see Figure 7.2.

7.2. The three-dimensional equations over a fixed domain; the fundamental scalings and assumptions on the data

We now describe the basic preliminaries of the asymptotic analysis of a linearly elastic shell, as set forth by Sanchez-Palencia (1990, 1992) in a slightly different, but in fact equivalent, framework of a 'multi-scale' asymptotic analysis, then by Miara and Sanchez-Palencia (1996), Ciarlet and Lods (1996*b*, 1996*d*), and Ciarlet, Lods and Miara (1996).

'Asymptotic analysis' means that the objective is to study the behaviour of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ as $\varepsilon \to 0$, an endeavour that will be achieved by studying the behaviour as $\varepsilon \to 0$ of the covariant components $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}$ of the displacement field, that is, the behaviour of the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ of the three-dimensional variational problem $\mathcal{P}(\Omega^\varepsilon)$ described above.

Since these fields are defined on sets $\overline{\Omega}^\varepsilon$ that themselves vary with ε, our first task naturally consists in transforming the three-dimensional problems $\mathcal{P}(\Omega^\varepsilon)$ into problems posed over a set that does not depend on ε. The underlying principle is thus identical to that followed for *plates*, albeit with differences in the way it is put to use (*cf.* Ciarlet (1997, Section 1.3)).

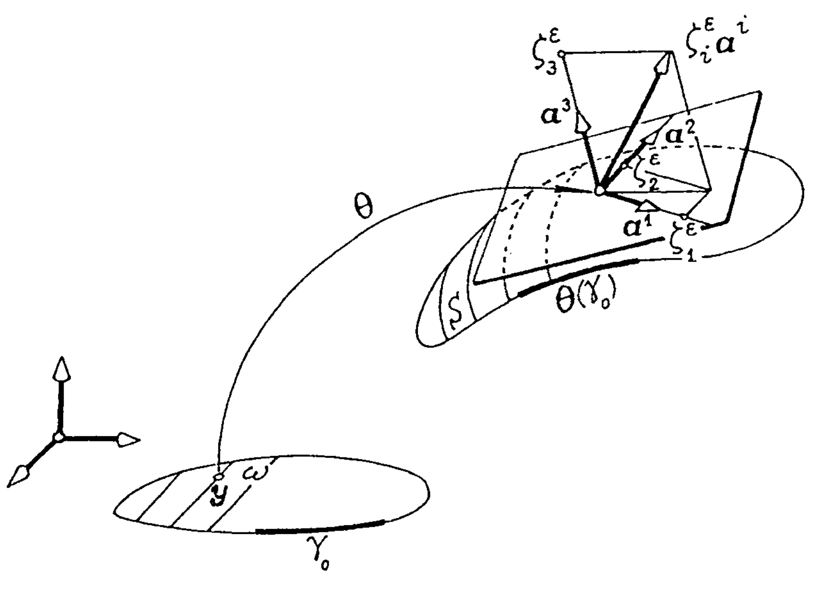

Fig. 7.2. A two-dimensional shell problem. The unknowns are the three covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the points of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. This means that, for each $y \in \overline{\omega}$, $\zeta_i^\varepsilon(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\theta}(y) \in S$

Let

$$\Omega := \omega \times]-1, 1[\text{ and } \Gamma_0 := \gamma_0 \times [-1, 1].$$

Let $x = (x_1, x_2, x_3)$ denote a generic point in the set $\overline{\Omega}$ and let $\partial_i := \partial/\partial x_i$; hence $x_\alpha = y_\alpha$, since a generic point in the set $\overline{\omega}$ is denoted by $y = (y_1, y_2)$. The coordinate $x_3 \in [-1, 1]$ will also be called *transverse variable*, like $x_3^\varepsilon \in [-\varepsilon, \varepsilon]$). With each point $x = (x_i) \in \overline{\Omega}$, we associate the point $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon$ through the bijection (see Figure 7.3)

$$\pi^\varepsilon : x = (x_1, x_2, x_3) \in \overline{\Omega} \longrightarrow x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon.$$

Note in passing that we therefore have $x_\alpha^\varepsilon = x_\alpha = y_\alpha$, $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$.

In order to carry out our asymptotic treatment of the solutions $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ of problems $\mathcal{P}(\Omega^\varepsilon)$ by considering ε as a small parameter, we must:

(i) specify the way the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ and more generally the vector fields $\boldsymbol{v} = (v_i^\varepsilon)$ appearing in the formulation of problems $\mathcal{P}(\Omega^\varepsilon)$ are mapped into vector fields over the set $\overline{\Omega}$;

(ii) control the way the Lamé constants and the applied body forces depend on the parameter ε.

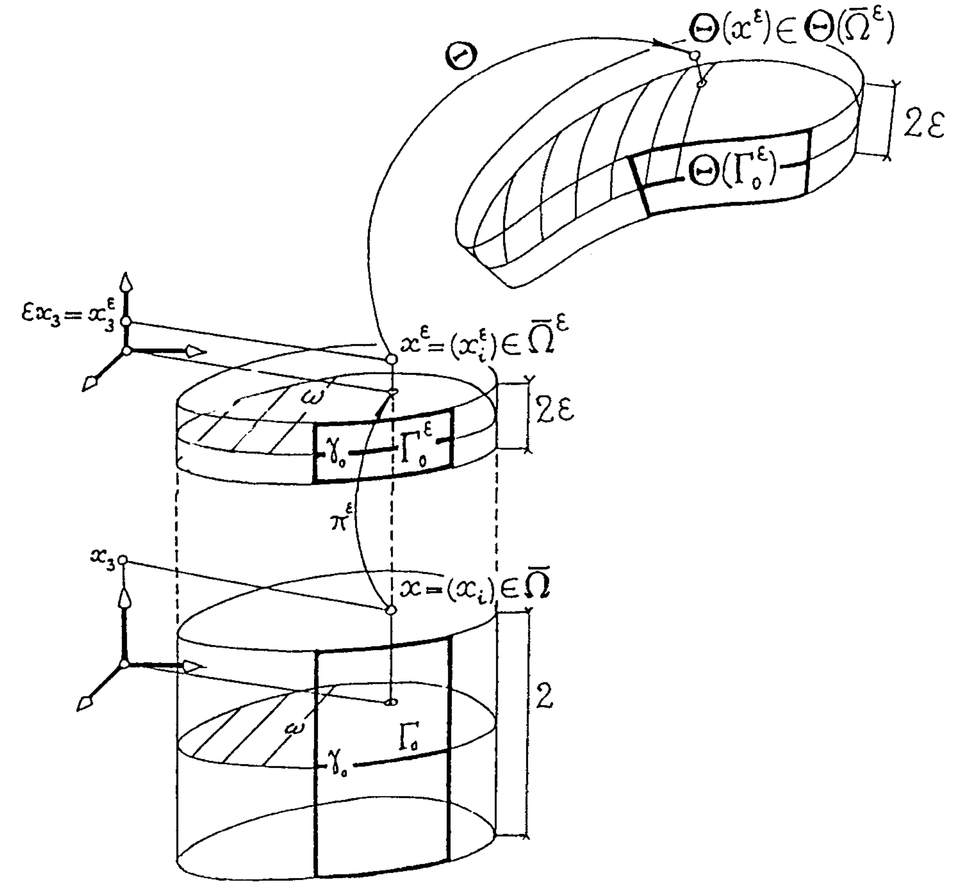

Fig. 7.3. Transformation of the three-dimensional shell problem into a 'scaled' problem, posed over the fixed domain $\Omega = \omega \times]-1, 1[$

With the unknown $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ and the vector fields $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ appearing in the three-dimensional variational problem $\mathcal{P}(\Omega^\varepsilon)$, we associate the *scaled unknown* $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ and the *scaled vector fields* $\boldsymbol{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$ defined by the *scalings*

$$u_i(\varepsilon)(x) := u_i^\varepsilon(x^\varepsilon) \text{ and } v_i(x) := v_i^\varepsilon(x^\varepsilon) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

The three components $u_i(\varepsilon)$ of the scaled unknown $\boldsymbol{u}(\varepsilon)$ are called the *scaled displacements.*

We next make the following assumptions on the data, that is, on the Lamé constants and on the applied body forces. There exist constants $\lambda > 0$ and $\mu > 0$ independent of ε and there exist functions $f^i \in L^2(\Omega)$ independent of

ε such that, for all $\varepsilon > 0$,

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^a f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon,$$

where the exponent a is for the time being left unspecified (needless to say, a is *not* subjected to the usual rule governing Latin exponents!)

Since the problem is linear, we assume without loss of generality that the scaled unknown $\boldsymbol{u}(\varepsilon)$ is 'of order 0 with respect to ε'. This means that the limit of $\boldsymbol{u}(\varepsilon)$ as ε approaches zero (assuming that such a limit exists, in an *ad hoc* function space) is *a priori* assumed to be of order 0, when the applied forces are of the right orders.

We have assumed that the Lamé constants are independent of ε. However, this assumption is merely a special case among a more general class of assumptions, which permit in particular the Lamé constants to vary with ε as $\varepsilon \to 0$ if one so wishes. More precisely, a multiplication of both Lamé constants by a factor $\varepsilon^t, t \in \mathbb{R}$, is always possible, as we shall see after Theorem 7.1. The choice $t = 0$ is merely made here for simplicity.

For sufficiently small $\varepsilon > 0$ (so that the mapping $\boldsymbol{\Theta}$ that defines the reference configuration of the shell is injective), a simple computation then produces the equations that the scaled unknown $\boldsymbol{u}(\varepsilon)$ satisfies over the set Ω, thus over a domain that is *independent of* ε (the Christoffel symbols $\Gamma^{3,\varepsilon}_{\alpha 3}$ and $\Gamma^{p,\varepsilon}_{33}$ vanish in Ω^ε for the special class of mappings $\boldsymbol{\Theta}$ considered here; consequently, the functions $\Gamma^3_{\alpha 3}(\varepsilon)$ and $\Gamma^p_{33}(\varepsilon)$ defined below likewise vanish in Ω).

Theorem 7.1: The three-dimensional shell problem over the fixed domain $\Omega = \omega \times]-1, 1[$. With the functions $\Gamma^{p,\varepsilon}_{ij}, g^\varepsilon, A^{ijkl,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbb{R}$ appearing in the equations of problem $\mathcal{P}(\Omega^\varepsilon)$, we associate the 'scaled' functions $\Gamma^p_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon) : \overline{\Omega} \to \mathbb{R}$ defined by

$$\Gamma^p_{ij}(\varepsilon)(x) := \Gamma^{p,\varepsilon}_{ij}(x^\varepsilon), \quad g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon), \quad A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon)$$

for all $x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon$.

With any vector field $\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)$, we associate the 'scaled linearized strains' $e_{i\|j}(\varepsilon; \boldsymbol{v}) = e_{j\|i}(\varepsilon; \boldsymbol{v}) \in L^2(\Omega)$ defined by

$$\begin{aligned}
e_{\alpha\|\beta}(\varepsilon; \boldsymbol{v}) &:= \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma^p_{\alpha\beta}(\varepsilon) v_p, \\
e_{\alpha\|3}(\varepsilon; \boldsymbol{v}) &:= \frac{1}{2}\left(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma^\sigma_{\alpha 3}(\varepsilon) v_\sigma, \\
e_{3\|3}(\varepsilon; \boldsymbol{v}) &:= \frac{1}{\varepsilon}\partial_3 v_3.
\end{aligned}$$

Let the assumptions on the data be as above. Then, for sufficiently small ε, the scaled unknown $\boldsymbol{u}(\varepsilon)$ satisfies the following scaled three-dimensional variational problem $\mathcal{P}(\varepsilon;\Omega)$ of a linearly elastic shell:

$$\boldsymbol{u}(\varepsilon) \in \mathbf{V}(\Omega) := \left\{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\right\},$$

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k\|l}(\varepsilon; \boldsymbol{u}(\varepsilon)) e_{i\|j}(\varepsilon; \boldsymbol{v}) \sqrt{g(\varepsilon)}\,\mathrm{d}x = \varepsilon^a \int_\Omega f^i v_i \sqrt{g(\varepsilon)}\,\mathrm{d}x$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$. □

The functions $A^{ijkl}(\varepsilon)$ are called the *contravariant components* of the scaled three-dimensional elasticity tensor of the shell. The functions $e_{i\|j}(\varepsilon;\boldsymbol{v})$ are called *'scaled' linearized strains* because they satisfy

$$e_{i\|j}(\varepsilon;\boldsymbol{v})(x) = e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

Note that the scaled strains $e_{i\|3}(\varepsilon;\boldsymbol{v})$ are *not* defined for $\varepsilon = 0$. Hence problems $\mathcal{P}(\varepsilon;\Omega)$ provide instances of singular perturbation problems in variational form, as considered and extensively studied by Lions (1973).

Note also that *exactly the same* scaled three-dimensional problem $\mathcal{P}(\varepsilon;\Omega)$ is evidently obtained if the scaled unknown is defined as before, but the following more general assumptions on the data are made:

$$\lambda^\varepsilon = \varepsilon^t\lambda \quad \text{and} \quad \mu^\varepsilon = \varepsilon^t\mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^{a+t} f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon,$$

where the constants $\lambda > 0$ and $\mu > 0$ and the functions $f^i \in L^2(\Omega)$ are as before independent of ε, but t is an *arbitrary real number.*

Our main objective now consists in establishing the *convergence* of the scaled unknown $\boldsymbol{u}(\varepsilon)$ in *ad hoc* function spaces as ε approaches zero. We shall see in this respect that there are essentially two distinct possible types of limit behaviour of $\boldsymbol{u}(\varepsilon)$, corresponding either to *linearly elastic 'membrane' shells* (Sections 8 and 9) or to *linearly elastic 'flexural' shells* (Section 10).

8. 'Elliptic membrane' shells

As we shall see, the classification of *linearly elastic shells* critically hinges on whether there exist nonzero displacement fields $\eta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}$ of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ that are both *linearized inextensional* ones, *i.e.*, that satisfy $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0$ in ω, and *admissible*, *i.e.*, that satisfy the boundary conditions $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0.

More specifically, define the space

$$\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0,$$
$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}.$$

Then a shell is called either a *linearly elastic 'membrane' shell* if $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$, that is, if $\mathbf{V}_F(\omega)$ contains only $\boldsymbol{\eta} = \mathbf{0}$, or a *linearly elastic 'flexural' shell* if $\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}$, that is, if $\mathbf{V}_F(\omega)$ contains *nonzero* elements.

A first instance where $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$ is provided by a *linearly elastic 'elliptic membrane' shell*, that is, one whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is *elliptic* (equivalently, the Gaussian curvature of S is everywhere > 0) and which is subjected to a boundary condition of place along its *entire* lateral face: that $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$ simply follows from the linearized rigid displacement lemma on an elliptic surface (Theorem 6.2).

The other instances where $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$ constitute the *linearly elastic 'generalized membrane' shells*, which will be studied in the next section.

The purpose of this section is to identify and to mathematically justify the *two-dimensional equations of a linearly elastic elliptic membrane shell*, by showing how the convergence of the three-dimensional displacements can be established in *ad hoc* function spaces as the thickness of such a shell approaches zero.

8.1. Definition and example

Let ω be a domain in $\mathbb{R}^2$ with boundary γ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\partial_\alpha\boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \overline{\omega}$. A shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is called a *linearly elastic elliptic membrane shell* if the following two conditions are simultaneously satisfied.

(i) The shell is subjected to a (homogeneous) *boundary condition of place* along its *entire* lateral face $\mathbf{\Theta}(\gamma \times [-\varepsilon,\varepsilon])$, that is, the displacement field vanishes there; equivalently,

$$\gamma_0 = \gamma.$$

(ii) The middle surface S is elliptic, in the sense that there exists a constant c such that

$$\sum_\alpha |\xi^\alpha|^2 \le c|b_{\alpha\beta}(y)\xi^\alpha\xi^\beta| \text{ for all } y \in \overline{\omega} \text{ and all } (\xi^\alpha) \in \mathbb{R}^2,$$

where the functions $b_{\alpha\beta} : \overline{\omega} \to \mathbb{R}$ are the covariant components of the curvature tensor of S (this definition was given in Section 6). This assumption means that the Gaussian curvature of S is everywhere > 0; equivalently, the two principal radii of curvature are either both > 0 at all points of S, or both < 0 at all points of S (see, *e.g.*, Ciarlet (2000, Section 2.2) for a detailed exposition of these notions).

A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of an ellipsoid, and which is subjected to a boundary condition of place, that is, of vanishing displacement field along its entire lateral face $\mathbf{\Theta}(\gamma \times [-\varepsilon,\varepsilon])$ (solid black in

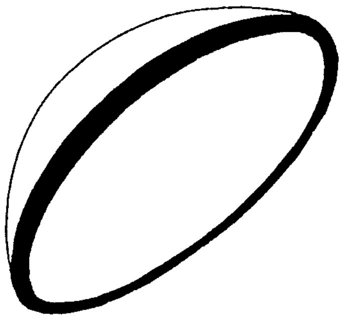

Fig. 8.1. A linearly elastic 'elliptic membrane' shell

the figure), provides an instance of a linearly elastic *elliptic membrane* shell; see Figure 8.1.

The definition of a linearly elastic elliptic membrane shell thus depends only on the subset of the lateral face where the shell is subjected to a boundary condition of place (via the set γ) and on the geometry of its middle surface.

If assumptions (i) and (ii) are satisfied and $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega}; \mathbb{R}^3)$, the linearized rigid displacement lemma on an elliptic surface (Theorem 6.2) shows that

$$\big\{\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega) : \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\} = \{\mathbf{0}\}.$$

Hence linearly elastic elliptic membrane shells indeed provide a first instance where the space

$$\begin{aligned}\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) :&\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\}&\end{aligned}$$

a fortiori reduces to $\{\mathbf{0}\}$. We recall that ∂_ν denotes the outer normal derivative operator along γ; the subscript 'F' reminds us that this space will be central to the study of *linearly elastic 'flexural' shells* in Section 10.

8.2. Convergence of the scaled displacements as the thickness approaches zero

We now establish the main results of this section. Consider a family of linearly elastic elliptic membrane shells with thickness $2\varepsilon > 0$ and with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, the assumption on the data being as in Theorem 8.1 below.

Then the solutions $\boldsymbol{u}(\varepsilon)$ of the associated scaled three-dimensional problems $\mathcal{P}(\varepsilon; \Omega)$ (Theorem 7.1) converge in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ as $\varepsilon \to 0$

toward a limit $\boldsymbol{u}$ and this limit, which is independent of the transverse variable x_3, can be identified with the solution $\overline{\boldsymbol{u}}$ of a two-dimensional variational problem $\mathcal{P}_M(\omega)$ posed over the set ω.

The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ appearing in the next theorem represent the covariant components of the *linearized change of metric tensor* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S.

Note that the assumption on the applied body forces made in the next theorem corresponds to letting $a = 0$ in Theorem 7.1. That $a = 0$ is indeed the 'correct' exponent in this case can be justified in two different ways:

It is easily checked that this choice is the only one that let the applied body forces enter (via the functions p^i) the right-hand sides of the variational equations in the 'limit' variational problem $\mathcal{P}_M(\omega)$ satisfied by $\overline{\boldsymbol{u}}$.

Otherwise, the number a can be considered *a priori* as an unknown. Then a *formal* asymptotic analysis of the scaled unknown $\boldsymbol{u}(\varepsilon)$ shows that, for a family of linearly elastic membrane shells (thus of the type considered here), the exponent a must be set equal to 0, again in order that the applied body forces contribute to the 'limit' variational problem; *cf.* Miara and Sanchez-Palencia (1996).

The following result is due to Ciarlet and Lods (1996*b*, Theorem 5.1); a complete proof is also given in Ciarlet (2000, Theorem 4.4-1).

Theorem 8.1: Convergence of the scaled displacements. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Consider a family of linearly elastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and assume that there exist constants $\lambda > 0$ and $\mu > 0$ and functions $f^i \in L^2(\Omega)$ independent of ε such that

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon.$$

(the notation is that of Section 7). Let $\boldsymbol{u}(\varepsilon)$ denote for sufficiently small $\varepsilon > 0$ the solution of the associated scaled three-dimensional problems $\mathcal{P}(\varepsilon; \Omega)$ (Theorem 7.1). Then there exist functions $u_\alpha \in H^1(\Omega)$ satisfying $u_\alpha = 0$ on $\gamma \times [-1, 1]$ and a function $u_3 \in L^2(\Omega)$ such that

$$u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(\Omega) \text{ and } u_3(\varepsilon) \to u_3 \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0,$$
$$\boldsymbol{u} := (u_i) \text{ is independent of the transverse variable } x_3.$$

Furthermore, the average $\overline{\boldsymbol{u}} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\,\mathrm{d}x_3$ satisfies the following two-dimensional variational problem $\mathcal{P}_M(\omega)$:

$$\overline{\boldsymbol{u}} = (\overline{u}_i) \in \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$
$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\overline{\boldsymbol{u}})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$. Here

$$\begin{aligned} \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \\ a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \\ p^i &:= \int_{-1}^{1} f^i \, \mathrm{d}x_3. \end{aligned}$$

Sketch of proof. **(i)** The proof rests on a crucial three-dimensional inequality of Korn's type for a family of linearly elastic elliptic membrane shells with each having the *same* elliptic middle surface $S = \boldsymbol{\theta}(\overline{\omega})$. For such a family, there exists a constant C such that, for sufficiently small $\varepsilon > 0$,

$$\left\{ \sum_\alpha \|v_\alpha\|^2_{1,\Omega} + |v_3|^2_{0,\Omega} \right\}^{1/2} \le C \left\{ \sum_{i,j} |e_{i\|j}(\varepsilon; \boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}$$

for all $\boldsymbol{v} = (v_i) \in \mathbf{V}(\Omega)$, where

$$\mathbf{V}(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \gamma \times [-1, 1]\},$$

and the functions $e_{i\|j}(\varepsilon; \boldsymbol{v})$ are the scaled linearized strains appearing in Theorem 7.1. Note that the proof of this inequality relies in a critical way on the inequality of Korn's type on an elliptic surface (Theorem 6.3).

(ii) The special form of the mapping $\boldsymbol{\Theta}$ that defines the reference configurations of the shells (Section 7) implies that there exists a constant C_e such that, for sufficiently small $\varepsilon > 0$,

$$\sum_{i,j} |t_{ij}|^2 \le C_e A^{ijkl}(\varepsilon)(x) t_{kl} t_{ij}$$

for all $x \in \overline{\Omega}$ and all symmetric matrices (t_{ij}).

Letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of problem $\mathcal{P}(\varepsilon; \Omega)$ (Theorem 7.1) and combining the three-dimensional inequality of Korn's type of (i) with the above inequality then yields a chain of inequalities implying that the norms $\|u_\alpha(\varepsilon)\|_{1,\Omega}$, $|u_3(\varepsilon)|_{0,\Omega}$, and $|e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon))|_{0,\Omega}$ are bounded independently of ε.

Thus there exists a subsequence, still denoted by $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ for notational convenience, such that

$$\begin{aligned} u_\alpha(\varepsilon) &\rightharpoonup u_\alpha \text{ in } H^1(\Omega), & u_\alpha(\varepsilon) &\to u_\alpha \text{ in } L^2(\Omega), \\ u_3(\varepsilon) &\rightharpoonup u_3 \text{ in } L^2(\Omega), & e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon)) &\rightharpoonup e_{i\|j} \text{ in } L^2(\Omega) \end{aligned}$$

(strong and weak convergence being respectively denoted by $\to$ and $\rightharpoonup$).

(iii) The above convergence, combined with the asymptotic behaviour of the functions $\Gamma^p_{ij}(\varepsilon)$, $A^{ijkl}(\varepsilon)$, and $g(\varepsilon)$, then implies that the functions u_i and $e_{i\|j}$ are independent of x_3 and that they are related by

$$\begin{aligned} e_{\alpha\|\beta} &= \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta} u_\sigma - b_{\alpha\beta} u_3, \\ e_{\alpha\|3} &= 0, \quad e_{3\|3} = -\frac{\lambda}{\lambda+2\mu} a^{\alpha\beta} e_{\alpha\|\beta}. \end{aligned}$$

(iv) In the variational equations of problem $\mathcal{P}(\varepsilon;\Omega)$, keep a function $\boldsymbol{v} \in \mathbf{V}(\Omega)$ fixed and let ε approach zero. Then the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$, combined with the relations found in (iii), shows that the average $\overline{\boldsymbol{u}} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\,\mathrm{d}x_3 \in \mathbf{V}_M(\omega)$ indeed satisfies the variational equations of the two-dimensional problem $\mathcal{P}_M(\omega)$ stated in the theorem.

The solution to $\mathcal{P}_M(\omega)$ being *unique*, the convergence established in (ii) for a subsequence thus holds for the *whole* family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$.

(v) Again letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of $\mathcal{P}(\varepsilon;\Omega)$ and using the results obtained in (ii)–(iv), we obtain the following *strong convergence*:

$$\begin{aligned} e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon)) &\to e_{i\|j} \text{ in } L^2(\Omega), \\ \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3 &\to \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\,\mathrm{d}x_3 \text{ in } H^1(\omega)\times H^1(\omega)\times L^2(\omega), \\ u_3(\varepsilon) &\to u_3 \text{ in } L^2(\omega). \end{aligned}$$

(vi) The strong convergence

$$u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(\Omega),$$

is then obtained as a consequence of the classical three-dimensional Korn inequality in Cartesian coordinates, combined with another use of the Lemma of J. L. Lions (Theorem 3.1). $\square$

8.3. The two-dimensional equations of a linearly elastic 'elliptic membrane' shell

The next theorem recapitulates the definition and assembles the main properties of the 'limit' two-dimensional variational problem $\mathcal{P}_M(\omega)$ found at the outcome of the asymptotic analysis carried out in Theorem 8.1. Note that $\mathcal{P}_M(\omega)$ is an *atypical variational problem*, in that one of the unknowns, namely, the third component ζ_3 of the vector field $\boldsymbol{\zeta} = (\zeta_i)$, 'only' lies in the space $L^2(\omega)$.

The existence and uniqueness theory, which is quickly reviewed in this theorem, is expounded in detail in Section 6, where *ad hoc* references are also provided.

Theorem 8.2: Existence, uniqueness, and regularity of solutions; formulation as a boundary value problem. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$ and such that the surface $S = \boldsymbol{\theta}(\overline{\omega})$ is elliptic.

(a) The associated two-dimensional variational problem $\mathcal{P}_M(\omega)$ found in Theorem 8.1 is as follows. Given functions $p^i \in L^2(\omega)$, find $\boldsymbol{\zeta} = (\zeta_i)$ satisfying

$$\boldsymbol{\zeta} \in \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$
$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$, where

$$\begin{aligned}\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,\\ a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big).\end{aligned}$$

This problem has exactly one solution, which is also the unique solution of the minimization problem:
Find $\boldsymbol{\zeta}$ such that

$$\boldsymbol{\zeta} \in \mathbf{V}_M(\omega) \ \text{ and } \ j_M(\boldsymbol{\zeta}) = \inf_{\boldsymbol{\eta}\in\mathbf{V}_M(\omega)} j_M(\boldsymbol{\eta}), \ \text{ where}$$
$$j_M(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y - \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y.$$

(b) If the solution $\boldsymbol{\zeta} = (\zeta_i)$ of $\mathcal{P}_M(\omega)$ is sufficiently smooth, it also satisfies the boundary value problem

$$\begin{aligned}-n^{\alpha\beta}|_\beta &= p^\alpha \text{ in } \omega,\\ -b_{\alpha\beta}n^{\alpha\beta} &= p^3 \text{ in } \omega,\\ \zeta_\alpha &= 0 \text{ on } \gamma,\end{aligned}$$

where

$$n^{\alpha\beta} := a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}) \ \text{ and } \ n^{\alpha\beta}|_\sigma := \partial_\sigma n^{\alpha\beta} + \Gamma^\alpha_{\sigma\tau}n^{\beta\tau} + \Gamma^\beta_{\sigma\tau}n^{\alpha\tau}.$$

(c) Assume that there exist an integer $m \geq 0$ and a real number $q > 1$ such that γ is of class $\mathcal{C}^{m+3}$, $\boldsymbol{\theta} \in \mathcal{C}^{m+3}(\overline{\omega};\mathbb{R}^3)$, $p^\alpha \in W^{m,q}(\omega)$, and $p^3 \in W^{m+1,q}(\omega)$. Then

$$\boldsymbol{\zeta} = (\zeta_i) \in W^{m+2,q}(\omega) \times W^{m+2,q}(\omega) \times W^{m+1,q}(\omega).$$

Proof. The existence and uniqueness of a solution to the variational problem $\mathcal{P}_M(\omega)$, or to its equivalent minimization problem, is a consequence of the inequality of Korn's type on an elliptic surface (Theorem 6.3), of the existence of constants c_e and a_0 such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ (Ciarlet 2000, Theorem 3.3-2) and $a(y) \geq a_0 > 0$ for all $y \in \overline{\omega}$, and of the Lax–Milgram lemma.

In view of finding the associated boundary value problem stated in part (b), we first note that

$$\partial_\alpha \sqrt{a} = \sqrt{a}\,\Gamma^\sigma_{\sigma\alpha},$$

as is easily verified.

Using Green's formula in Sobolev space and assuming that the functions $n^{\alpha\beta} = n^{\beta\alpha}$ are in $H^1(\omega)$, we next obtain

$$\begin{aligned}
&\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\,\mathrm{d}y = \int_\omega n^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\,\mathrm{d}y \\
&= \int_\omega \sqrt{a}\, n^{\alpha\beta} \left(\frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3 \right) \mathrm{d}y \\
&= \int_\omega \sqrt{a}\, n^{\alpha\beta} \partial_\beta \eta_\alpha \,\mathrm{d}y - \int_\omega \sqrt{a}\, n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma \,\mathrm{d}y - \int_\omega \sqrt{a}\, n^{\alpha\beta} b_{\alpha\beta} \eta_3 \,\mathrm{d}y \\
&= -\int_\omega \partial_\beta \big(\sqrt{a}\, n^{\alpha\beta}\big) \eta_\alpha \,\mathrm{d}y - \int_\omega \sqrt{a}\, n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma \,\mathrm{d}y - \int_\omega \sqrt{a}\, n^{\alpha\beta} b_{\alpha\beta} \eta_3 \,\mathrm{d}y \\
&= -\int_\omega \sqrt{a} \big(\partial_\beta n^{\alpha\beta} + \Gamma^\alpha_{\tau\beta} n^{\tau\beta} + \Gamma^\beta_{\beta\tau} n^{\alpha\tau}\big) \eta_\alpha \,\mathrm{d}y - \int_\omega \sqrt{a}\, n^{\alpha\beta} b_{\alpha\beta} \eta_3 \,\mathrm{d}y \\
&= -\int_\omega \sqrt{a} \big\{ \big(n^{\alpha\beta}|_\beta\big) \eta_\alpha + b_{\alpha\beta} n^{\alpha\beta} \eta_3 \big\} \,\mathrm{d}y
\end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$. Hence the variational equations imply that

$$\int_\omega \sqrt{a} \big\{ \big(n^{\alpha\beta}|_\beta + p^\alpha\big) \eta_\alpha + \big(b_{\alpha\beta} n^{\alpha\beta} + p^3\big) \eta_3 \big\} \,\mathrm{d}y = 0$$

for all $(\eta_i) \in \mathbf{V}_M(\omega)$, and thus $n^{\alpha\beta}|_\beta = p^\alpha$ and $b_{\alpha\beta} n^{\alpha\beta} = p^3$ in ω.

The regularity result of part (c) is due to Genevey (1996). $\square$

Note that the functions $n^{\alpha\beta}|_\sigma$ are instances of *first-order covariant derivatives of a tensor field*, defined here by means of its contravariant components $n^{\alpha\beta}$.

In order to get physically meaningful formulas, it remains to 'de-scale' the unknowns ζ_i that satisfy the limit 'scaled' problem $\mathcal{P}_M(\omega)$ found in Theorem 8.2. In view of the scalings $u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon)$ for all $x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon$ made on the covariant components of the displacement field (Section 7), we are led to defining for each $\varepsilon > 0$ the covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the 'limit displacement field' $\zeta_i^\varepsilon \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface S of the shell by letting

$$\zeta_i^\varepsilon := \zeta_i$$

(the vectors $\boldsymbol{a}^i$ forming the contravariant basis at each point of S).

Naturally, the field (ζ_i^ε) and the field $\zeta_i^\varepsilon \boldsymbol{a}^i$ must be carefully distinguished! The former is essentially a convenient mathematical 'intermediary', while only the latter has physical significance.

Recall that $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ represent the contravariant components of the applied body forces *actually* acting on the shell and that λ^ε and μ^ε denote the *actual* Lamé constants of its constituent material. We then have the following immediate corollary to Theorems 8.1 and 8.2. Naturally, the existence, uniqueness and regularity results of Theorem 8.2 apply *verbatim* to the solution of the 'de-scaled' problem $\mathcal{P}_M^\varepsilon(\omega)$ found in the next theorem (for this reason, they are not reproduced here).

Theorem 8.3: The two-dimensional equations of a linearly elastic 'elliptic membrane' shell. Let the assumptions on the data be as in Theorem 8.1. Then the vector field $\boldsymbol{\zeta}^\varepsilon := (\zeta_i^\varepsilon)$ formed by the covariant components of the limit displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i$ of the middle surface S satisfies the following two-dimensional variational problem $\mathcal{P}_M^\varepsilon(\omega)$ of a linearly elastic elliptic membrane shell:

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

$$\varepsilon \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y = \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_M(\omega)$, where

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - b_{\alpha\beta}\eta_3,\\
a^{\alpha\beta\sigma\tau,\varepsilon} &:= \frac{4\lambda^\varepsilon \mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^\varepsilon (a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),\\
p^{i,\varepsilon} &:= \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\, \mathrm{d}x_3^\varepsilon.
\end{aligned}$$

Equivalently, the field $\boldsymbol{\zeta}^\varepsilon$ satisfies the minimization problem

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_M(\omega) \ \text{ and } \ j_M^\varepsilon(\boldsymbol{\zeta}^\varepsilon) = \inf_{\boldsymbol{\eta} \in \mathbf{V}_M(\omega)} j_M^\varepsilon(\boldsymbol{\eta}), \ \text{where}$$

$$j_M^\varepsilon(\boldsymbol{\eta}) := \frac{\varepsilon}{2} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y - \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y.$$

If the field $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$ is sufficiently smooth, it also satisfies the following boundary value problem:

$$-n^{\alpha\beta,\varepsilon}|_\beta = p^{\alpha,\varepsilon} \text{ in } \omega,$$
$$-b_{\alpha\beta} n^{\alpha\beta,\varepsilon} = p^{3,\varepsilon} \text{ in } \omega,$$
$$\zeta_\alpha^\varepsilon = 0 \text{ on } \gamma,$$

where

$$n^{\alpha\beta,\varepsilon} := \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon),$$
$$n^{\alpha\beta,\varepsilon}|_\sigma := \partial_\sigma n^{\alpha\beta,\varepsilon} + \Gamma^\alpha_{\tau\sigma} n^{\tau\beta,\varepsilon} + \Gamma^\beta_{\sigma\tau} n^{\alpha\tau,\varepsilon}. \qquad \square$$

Each one of the three formulations found in Theorem 8.3 constitutes one version of the two-dimensional equations of a linearly elastic elliptic membrane shell. The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ are the *covariant components* of the *linearized change of metric tensor* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S, the functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the *contravariant components* of the *two-dimensional elasticity tensor of the shell*, and the functions $n^{\alpha\beta,\varepsilon}$ are the contravariant components of the *stress resultant tensor* field.

The functional $j_M^\varepsilon : \mathbf{V}_M(\omega) \to \mathbb{R}$ is the *two-dimensional energy*, and the functional

$$\boldsymbol{\eta} \in \mathbf{V}_M(\omega) \to \frac{\varepsilon}{2} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, \mathrm{d}y$$

is the *two-dimensional strain energy* of a *linearly elastic elliptic membrane shell*.

Finally, the equations $-n^{\alpha\beta,\varepsilon}|_\beta = p^{\alpha,\varepsilon}$ and $-b_{\alpha\beta} n^{\alpha\beta,\varepsilon} = p^{3,\varepsilon}$ in ω constitute the *two-dimensional equations of equilibrium*, and the relations $n^{\alpha\beta,\varepsilon} = \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)$ constitute the *two-dimensional constitutive equation* of a *linearly elastic elliptic membrane shell.*

Under the essential assumptions that $\gamma_0 = \gamma$ and that the surface S is elliptic, we have therefore justified by a convergence result (Theorem 8.1) two-dimensional equations that are called those of a linearly elastic 'membrane' shell in the literature (which, however, usually ignores the distinction between 'elliptic' and 'generalized' membrane shells); see, *e.g.*, Koiter (1966, equations (9.14) and (9.15)), Green and Zerna (1968, Section 11.1), Dikmen (1982, equations (7.10)), or Niordson (1985, equation (10.3)).

In so doing, we have also justified the formal asymptotic approach of Sanchez-Palencia (1990) (see also Miara and Sanchez-Palencia (1996), Caillerie and Sanchez-Palencia (1995*b*), Faou (2000*a*, 2000*b*)) when 'bending is well-inhibited', according to the terminology of E. Sanchez-Palencia.

Note that the above convergence analysis also substantiates an important observation. In an *elliptic membrane shell*, body forces of order $O(1)$ with respect to ε produce a limit displacement field that is also $O(1)$. By contrast, body forces must be of order $O(\varepsilon^2)$ in order to produce an $O(1)$ limit displacement field in a *flexural shell*. See Section 10.

After the original work of Ciarlet and Lods (1996*b*) described in this section, the asymptotic analysis of linearly elastic membrane shells underwent several refinements and generalizations.

First, Genevey (1999) has shown that the convergence result of Theorem 8.1 can also be obtained by resorting to Γ-convergence theory.

Using the techniques of Lions (1973), Mardare (1998*a*) was able to compute a *corrector*, so as to obtain in this fashion the following remarkable error estimate. In addition to the hypotheses made in Theorem 8.1, assume that the boundary of the domain ω is of class $\mathcal{C}^2$, that $\partial_\alpha f^\alpha \in L^2(\Omega)$, and that

$$\boldsymbol{\zeta} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, \mathrm{d}x_3 \in \mathbf{H}^2(\omega) \cap \mathbf{V}_M(\omega).$$

Then there exists a constant $C = C(\omega, \boldsymbol{\theta}, f^i, \boldsymbol{\zeta})$ independent of ε such that

$$\|\boldsymbol{u}(\varepsilon) - \boldsymbol{u}\|_{H^1(\Omega)\times H^1(\Omega)\times L^2(\Omega)} \leq \mathcal{C}\varepsilon^{1/6},$$

and moreover, the exponent $1/6$ is the best possible.

Other useful extensions include the justification by an asymptotic analysis of linearly elastic membrane shells with variable thickness (Busse 1998), the convergence of the stresses and the explicit forms of the limit stresses (Collard and Miara 1999), an asymptotic analysis of the associated time-dependent problem (Xiao Li-Ming 1998), and the extension of the present analysis to shells whose middle surface is elliptic but has 'no boundary', such as an entire ellipsoid (Ramos (1995) and Şlicaru (1997)).

The variational formulation of the limit two-dimensional problem of a linearly elastic elliptic membrane shell (Theorem 8.3) possesses the unusual feature that its third unknown ζ_3^ε 'only' belongs to the space $L^2(\omega)$. This explains why the averaged three-dimensional boundary condition

$$\overline{u}_3^\varepsilon := \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \,\mathrm{d}x_3^\varepsilon = 0 \text{ on } \gamma$$

is 'lost' as $\varepsilon \to 0$, since $\zeta_3^\varepsilon = 0$ on γ does not make sense. As expected, this loss is compensated by the appearance of a *boundary layer* in the unknown ζ_3^ε.

Again because the third unknown ζ_3^ε is only in $L^2(\omega)$, the linear operator associated with the variational problem of a linearly elastic elliptic membrane shell is *not* compact and thus the analysis of the corresponding eigenvalue problem requires special care; see Sanchez-Hubert and Sanchez-Palencia (1997, Chapter 10).

9. 'Generalized membrane' shells

A shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, subjected to a boundary condition of place along a portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$, where $\gamma_0 \subset \gamma$, as its middle curve, is a *linearly elastic 'generalized membrane' shell* if it is *not* an elliptic membrane shell according to the definition given in Section 8, yet if its associated space

$$\mathbf{V}_F(\omega) = \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\}$$

still reduces to $\{\mathbf{0}\}$. As shown in Section 9.1, examples of generalized membrane shells abound.

The purpose of this section is to identify and to mathematically justify the *two-dimensional equations of a linearly elastic generalized membrane shell*, by establishing the convergence of the three-dimensional displacements in *ad hoc* function spaces as the thickness of such a shell approaches zero.

9.1. Definition and examples

Let ω be a domain in $\mathbb{R}^2$ with boundary γ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \overline{\omega}$. A shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is called a *linearly elastic generalized membrane shell* if the following three conditions are simultaneously satisfied.

(i) The shell is subjected to a (homogeneous) *boundary condition of place* (*i.e.*, of vanishing displacement field) along a portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, where the subset $\gamma_0 \subset \gamma$ satisfies

$$\text{length } \gamma_0 > 0.$$

(ii) Define the space

$$\mathbf{V}_F(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\}$$

(∂_ν denoting the outer normal derivative operator along γ). Then the space $\mathbf{V}_F(\omega)$ contains only $\boldsymbol{\eta} = \mathbf{0}$.

(iii) The shell is not an elliptic membrane shell. We recall that a linearly elastic shell is an 'elliptic membrane' shell if $\gamma_0 = \gamma$ and S is elliptic (Section 8.1) and that an elliptic membrane shell also provides an instance where the space $\mathbf{V}_F(\omega)$, which in this case is the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$, reduces to $\{\mathbf{0}\}$ (at least if $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\omega; \mathbb{R}^3)$; *cf.* Theorem 6.2). Generalized membrane shells thus exhaust all the remaining cases of linearly elastic membrane shells, *i.e.*, those for which $\mathbf{V}_F(\omega) = \{\mathbf{0}\}$.

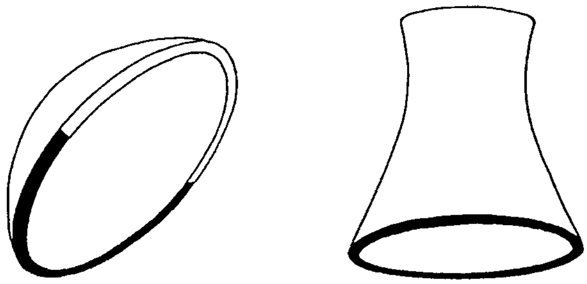

Fig. 9.1. Two examples of linearly elastic 'generalized membrane' shells

The definition of a linearly elastic 'generalized membrane' shell thus depends only on the subset of the lateral face where the shell is subjected to a boundary condition of place (via the set γ_0) and on the geometry of the middle surface of the shell.

As shown by Vekua (1962) under the assumptions that $\boldsymbol{\theta} \in W^{3,p}(\omega; \mathbb{R}^3)$ for some $p > 2$ and that γ is of class $\mathcal{C}^3$, then by Lods and Mardare (1998a) under the assumption that $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega}; \mathbb{R}^3)$ and that γ is Lipschitz-continuous, a shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of an ellipsoid and which is subjected to a boundary condition of place along a portion (solid black in the figure) of its lateral face whose middle curve $\boldsymbol{\theta}(\gamma_0)$ is such that $0 <$ length $\gamma_0 <$ length γ, provides an instance of a linearly elastic generalized membrane shell; see Figure 9.1. A comparison with Figure 8.1 illustrates the crucial role played by the set $\boldsymbol{\theta}(\gamma_0)$ in determining the type of shell!

As shown by Mardare (1998c) under the assumption that $\boldsymbol{\theta} \in \mathcal{C}^{2,1}(\overline{\omega}; \mathbb{R}^3)$ (see also Vekua (1962) and Sanchez-Hubert and Sanchez-Palencia (1997, Chapter 7, Section 2)), a shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a hyperboloid of revolution and which is subjected to a boundary condition of place along its entire 'lower' lateral face provides another instance of a linearly elastic generalized membrane shell; see Figure 9.1.

A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a cone or a cylinder and which is subjected to a boundary condition of place along a portion (solid black in the figure) of its lateral face with $\boldsymbol{\theta}(\gamma_0)$ as its middle curve is a linearly elastic generalized membrane shell if $\boldsymbol{\theta}(\gamma_0)$ intersects all the generatrices of S; see Figure 9.2.

As for elliptic membrane shells (Section 8), the formal asymptotic analysis of Miara and Sanchez-Palencia (1996) again suggests making the following assumptions on the data for a family of generalized membrane shells. We require that the Lamé constants and the applied body densities appearing

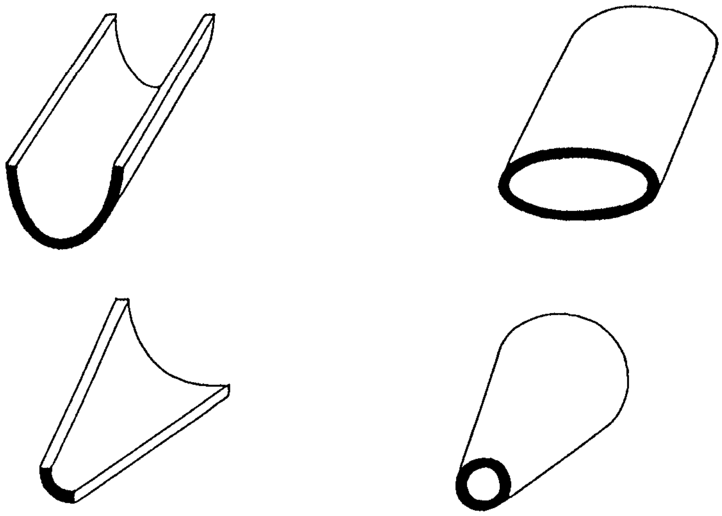

Fig. 9.2. Other examples of linearly elastic 'generalized membrane' shells

in the three-dimensional problems $\mathcal{P}(\Omega^\varepsilon)$ (Section 7) satisfy

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon,$$

where the constants $\lambda > 0$ and $\mu > 0$ and the functions $f^i \in L^2(\Omega)$ are independent of ε. In other words, the exponent a in Theorem 7.1 again vanishes.

It turns out, however, that in order to carry out the asymptotic analysis of generalized membrane shells, we have to make a specific, and rather stringent, *assumption on the applied forces*, which supersedes in fact the above one, in such a way that the linear form appearing in the variational problem $\mathcal{P}(\varepsilon;\Omega)$ of Theorem 7.1 becomes continuous with respect to an *ad hoc* norm, and uniformly so with respect to ε. We now describe this assumption, particular to generalized membrane shells.

9.2. 'Admissible' applied forces

Consider a *family of linearly elastic generalized membrane shells*, with thickness 2ε, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, and let the assumptions on the data be as in Section 9.1.

Let

$$\mathbf{V}(\Omega) := \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\},$$

and, for each $\varepsilon > 0$, let the linear form $L(\varepsilon) : \mathbf{V}(\Omega) \to \mathbb{R}$ be defined by

$$L(\varepsilon)(\boldsymbol{v}) := \int_\Omega f^i v_i \sqrt{g(\varepsilon)}\,\mathrm{d}x \quad \text{for all} \quad \boldsymbol{v} \in \mathbf{V}(\Omega).$$

In other words, $L(\varepsilon)(\boldsymbol{v})$ is the right-hand side in problem $\mathcal{P}(\varepsilon;\Omega)$ (Theorem 7.1), which takes into account the applied body forces through the functions $f^i \in L^2(\Omega)$. Then each linear form $L(\varepsilon) : \mathbf{V}(\Omega) \to \mathbb{R}$ is clearly continuous with respect to the norm $\|\cdot\|_{1,\Omega}$ and uniformly so with respect to sufficiently small $\varepsilon > 0$.

It so happens, however, that an essentially stronger property is needed. The linear forms $L(\varepsilon)$ should also be continuous, and uniformly so, with respect to sufficiently small $\varepsilon > 0$, and with respect to the norm (itself dependent on ε)

$$\boldsymbol{v} \to \left\{ \sum_{i,j} |e_{i\|j}(\varepsilon;\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}.$$

In order to fulfil this requirement in a concrete manner, we set the following definition. Applied forces acting on a family of linearly elastic generalized membrane shells are said to be 'admissible' if there exist functions $F^{ij}(\varepsilon) = F^{ji}(\varepsilon) \in L^2(\Omega)$ and functions $F^{ij} = F^{ji} \in L^2(\Omega)$ such that

$$L(\varepsilon)(\boldsymbol{v}) = \int_\Omega F^{ij}(\varepsilon) e_{i\|j}(\varepsilon;\boldsymbol{v}) \sqrt{g(\varepsilon)}\,\mathrm{d}x$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, and

$$F^{ij}(\varepsilon) \to F^{ij} \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

If the applied forces are admissible, there thus exists a constant κ_0 such that

$$|L(\varepsilon)(\boldsymbol{v})| \leq \kappa_0 \left\{ \sum_{i,j} |e_{i\|j}(\varepsilon;\boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}$$

for all $0 < \varepsilon \leq \varepsilon_0$ and for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, as was required.

This inequality will be put to an essential use in Theorem 9.1 for finding the *a priori* bounds that the family of scaled unknowns satisfies.

The convergence $F^{ij}(\varepsilon) \to F^{ij}$ in $L^2(\Omega)$ serves a further purpose, that of defining the right-hand sides appearing in the 'limit' two-dimensional problems (see again Theorem 9.1).

Naturally, admissible forces have to be identified for each instance of generalized membrane shells; see in this respect the references given in Section 9.4.

9.3. Convergence of the scaled displacements as the thickness approaches zero

A generalized membrane shell is 'of the first kind' if the space

$$\mathbf{V}_0(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\},$$

which is larger than the space $\mathbf{V}_F(\omega)$, 'already' reduces to $\{\mathbf{0}\}$. Equivalently, the seminorm $|\cdot|_\omega^M$ defined by

$$|\boldsymbol{\eta}|_\omega^M = \left\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|_{0,\omega}^2\right\}^{1/2}$$

is 'already' a norm over the space

$$\mathbf{V}(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\}.$$

As *all* the known examples of generalized membrane shells satisfy this assumption, we shall not consider here the generalized membrane shells 'of the second kind', *i.e.*, those for which $\mathbf{V}_F(\omega)$ contains only $\boldsymbol{\eta} = \mathbf{0}$, but $\mathbf{V}_0(\omega)$ contains *nonzero* elements. Such shells are analysed in Ciarlet and Lods (1996*d*, Theorem 5.1); see also Ciarlet (2000, Theorem 5.6-2).

We now establish the main results of this section. Consider a *family of linearly elastic generalized membrane shells of the first kind*, with thickness $2\varepsilon > 0$, with each having the *same* middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the *same* set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, the applied forces being admissible. Then the averages

$$\overline{\boldsymbol{u}(\varepsilon)} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3$$

of the scaled unknowns converge in an 'abstract' completion $\mathbf{V}_M^\sharp(\omega)$ as $\varepsilon \to 0$ and their limit satisfies an 'abstract' variational problem posed over the same space $\mathbf{V}_M^\sharp(\omega)$.

The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ appearing in the next theorem represent the covariant components of the *linearized change of metric tensor* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the surface S. Hence the bilinear form B_M defined below coincides with that found in the scaled variational problem of a linearly elastic elliptic membrane shell (Theorem 8.2).

The following result is due to Ciarlet and Lods (1996*d*, Theorem 5.1); a complete proof is also given in Ciarlet (2000, Theorem 5.6-1). In these references, it is also shown how the convergence of the scaled unknowns $\boldsymbol{u}(\varepsilon)$ themselves can be also established, in an *ad hoc* completion.

Theorem 9.1: Convergence of the scaled displacement. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Consider a family of linearly elastic generalized membrane shells of the first kind, with thickness 2ε approaching zero, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve. Assume that there exist constants $\lambda > 0$ and $\mu > 0$ such that

$$\lambda^\varepsilon = \lambda \text{ and } \mu^\varepsilon = \mu.$$

Finally, assume that the applied forces are admissible (Section 9.2). For sufficiently small $\varepsilon > 0$, let $\boldsymbol{u}(\varepsilon)$ denote the solution of the associated scaled three-dimensional problems $\mathcal{P}(\varepsilon;\Omega)$ (Theorem 7.1).

Define the space

$$\mathbf{V}_M^\sharp(\omega) := \text{completion of } \mathbf{V}(\omega) \text{ with respect to } |\cdot|_\omega^M.$$

Then there exists $\boldsymbol{\zeta} \in \mathbf{V}_M^\sharp(\omega)$ such that

$$\overline{\boldsymbol{u}(\varepsilon)} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon)\,\mathrm{d}x_3 \to \boldsymbol{\zeta} \quad \text{in } \mathbf{V}_M^\sharp(\omega) \quad \text{as } \varepsilon \to 0.$$

Let

$$\begin{aligned}
a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),\\
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,\\
B_M(\boldsymbol{\zeta},\boldsymbol{\eta}) &:= \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\zeta},\boldsymbol{\eta} \in \mathbf{V}(\omega),\\
L_M(\boldsymbol{\eta}) &:= \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\eta} \in \mathbf{V}(\omega),\\
\varphi^{\alpha\beta} &:= \int_{-1}^{1}\left\{F^{\alpha\beta} - \frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}F^{33}\right\}\mathrm{d}x_3 \in L^2(\omega),
\end{aligned}$$

where the functions $F^{ij} \in L^2(\Omega)$ are those used in the definition of admissible forces, and let $B_M^\sharp$ and $L_M^\sharp$ denote the unique continuous extensions from $\mathbf{V}(\omega)$ to $\mathbf{V}_M^\sharp(\omega)$ of the bilinear form B_M and linear form L_M. Then the limit $\boldsymbol{\zeta}$ satisfies the following two-dimensional variational problem $\mathcal{P}_M^\sharp(\omega)$:

$$\boldsymbol{\zeta} \in \mathbf{V}_M^\sharp(\omega) \text{ and } B_M^\sharp(\boldsymbol{\zeta},\boldsymbol{\eta}) = L_M^\sharp(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_M^\sharp(\omega).$$

Sketch of proof. **(i)** The proof rests on a crucial three-dimensional inequality of Korn's type for a family of linearly elastic shells, with each having the *same* middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the *same* set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve. For such a family, there exists a constant C such that, for sufficiently small $\varepsilon > 0$,

$$\|\boldsymbol{v}\|_{1,\Omega} \leq \frac{C}{\varepsilon}\left\{\sum_{i,j} |e_{i\|j}(\varepsilon;\boldsymbol{v})|_{0,\Omega}^2\right\}^{1/2}$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, where

$$\mathbf{V}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \gamma_0 \times [-1,1]\},$$

and the functions $e_{i\|j}(\varepsilon;\boldsymbol{v})$ are the scaled linearized strains appearing in Theorem 7.1. The proof of this inequality relies in a critical way on the linearized rigid displacement lemma on a general surface (Theorem 4.3).

(ii) Given a function $v \in L^2(\Omega)$, let

$$\overline{v} := \frac{1}{2}\int_{-1}^{1} v \,\mathrm{d}x_3 \in L^2(\omega)$$

denote its *average* with respect to the transverse variable x_3; the same notation is used for vector-valued functions. Letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of problem $\mathcal{P}(\varepsilon;\Omega)$ (Theorem 7.1) and using the three-dimensional inequality of Korn's type of (i) then yields a chain of inequalities showing that the norms $|\partial_3\boldsymbol{u}(\varepsilon)|_{0,\Omega}$, $|\overline{\boldsymbol{u}(\varepsilon)}|_\omega^M$, $|e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon)|_{0,\Omega}$, and $\|\varepsilon\boldsymbol{u}(\varepsilon)\|_{1,\Omega}$ are bounded independently of ε. Note that the assumption that the applied forces are 'admissible' is crucial here.

Thus there exists a subsequence, still denoted by $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ for notational convenience, such that $\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u}$ in the completion of the space $\mathbf{V}(\Omega)$ with respect to the norm $|\cdot|_\Omega^M$ defined by $|\boldsymbol{v}|_\Omega^M := \{|\partial_3\boldsymbol{v}|_{0,\Omega}^2 + (|\overline{\boldsymbol{v}}|_\omega^M)^2\}^{1/2}$, and such that

$$\begin{aligned} e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon)) &\rightharpoonup e_{i\|j} \text{ in } L^2(\Omega), & \varepsilon\boldsymbol{u}(\varepsilon) &\rightharpoonup \boldsymbol{u}^{-1} \text{ in } \mathbf{H}^1(\Omega),\\ \partial_3 u_3(\varepsilon) = \varepsilon e_{33}(\varepsilon;\boldsymbol{u}(\varepsilon)) &\to 0 \text{ in } L^2(\Omega), & \overline{\boldsymbol{u}(\varepsilon)} &\rightharpoonup \boldsymbol{\zeta} \text{ in } \mathbf{V}_M^\sharp(\omega).\end{aligned}$$

(iii) The above convergence, combined with the asymptotic behaviour of the functions $\Gamma_{ij}^p(\varepsilon), A^{ijkl}(\varepsilon)$, and $g(\varepsilon)$, then implies that

$$\begin{aligned} e_{\alpha\|3} &= \frac{1}{2\mu}a_{\alpha\beta}F^{\beta 3},\\ e_{3\|3} &:= -\frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}e_{\alpha\|\beta} + \frac{F^{33}}{\lambda+2\mu},\\ \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) &\rightharpoonup \overline{e_{\alpha\|\beta}} \text{ in } L^2(\omega),\\ \varepsilon\boldsymbol{u}(\varepsilon) &\rightharpoonup 0 \text{ in } \mathbf{H}^1(\Omega),\\ \partial_3 u_\alpha(\varepsilon) &\rightharpoonup 0 \text{ in } L^2(\omega),\\ e_{\alpha\|\beta} &\text{ is independent of the transverse variable } x_3,\end{aligned}$$

where the functions $F^{ij} \in L^2(\Omega)$ are those appearing in the definition of 'admissible' forces.

(iv) In the variational equations of problem $\mathcal{P}(\varepsilon;\Omega)$, let $\boldsymbol{v} \in \mathbf{V}(\Omega)$ be independent of the transverse variable x_3. Keep such a function $\boldsymbol{v}$ *fixed* and let ε approach zero. Then the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$ combined with the relations found in (ii) and (iii) together show that the limits $e_{\alpha\|\beta}$ found in part (ii) satisfy

$$\int_\omega a^{\alpha\beta\sigma\tau}\overline{e_{\sigma\|\tau}}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),$$

where

$$\varphi^{\alpha\beta} := \int_{-1}^{1}\left\{F^{\alpha\beta} - \frac{\lambda}{\lambda+2\mu}a^{\alpha\beta}F^{33}\right\}\mathrm{d}x_3 \in L^2(\omega).$$

(v) Again letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of $\mathcal{P}(\varepsilon;\Omega)$ and using the results obtained in (ii)–(iv), we obtain the following strong convergence:

$$\begin{aligned} e_{i\|j}(\varepsilon;\boldsymbol{u}(\varepsilon)) &\to e_{i\|j} \text{ in } L^2(\Omega),\\ \varepsilon\boldsymbol{u}(\varepsilon) &\to \mathbf{0} \text{ in } \mathbf{H}^1(\Omega),\\ \gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) &\to \overline{e_{\alpha\|\beta}} \text{ in } L^2(\omega),\\ \overline{\boldsymbol{u}(\varepsilon)} &\to \boldsymbol{\zeta} \text{ in } \mathbf{V}^\sharp_M(\omega). \end{aligned}$$

(vi) The convergence $\gamma_{\alpha\beta}(\overline{\boldsymbol{u}(\varepsilon)}) \to \overline{e_{\alpha\|\beta}}$ in $L^2(\omega)$ implies that the limit $\boldsymbol{\zeta} \in \mathbf{V}^\sharp_M(\omega)$ found in (v) satisfies the equations

$$B^\sharp_M(\boldsymbol{\zeta},\boldsymbol{\eta}) = L^\sharp_M(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}^\sharp_M(\omega),$$

which have a unique solution. Consequently, the convergence

$$\overline{\boldsymbol{u}(\varepsilon)} \to \boldsymbol{\zeta} \text{ in } \mathbf{V}^\sharp_M(\omega)$$

established in (v) holds for the *whole* family $(\overline{\boldsymbol{u}(\varepsilon)})_{\varepsilon>0}$. □

9.4. The two-dimensional equations of a linearly elastic 'generalized membrane' shell

Again, we only consider generalized membrane shells of the first kind. The next theorem recapitulates the definition and assembles the main properties of the 'limit' two-dimensional problem found at the outcome of the asymptotic analysis carried out in Theorem 9.1.

Theorem 9.2: Existence and uniqueness of solutions. Let ω be a domain in $\mathbb{R}^2$, let γ_0 be a subset of the boundary of ω with length $\gamma_0 > 0$, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors

$\boldsymbol{a}_1 = \partial_1\boldsymbol{\theta}, \boldsymbol{a}_2 = \partial_2\boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$. Assume that $\mathbf{V}_0(\omega) = \{\mathbf{0}\}$, where

$$\begin{aligned}\mathbf{V}_0(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\},\\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3,\end{aligned}$$

and define the spaces

$$\begin{aligned}\mathbf{V}(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\},\\ \mathbf{V}^\sharp_M(\omega) &:= \text{completion of } \mathbf{V}(\omega) \text{ with respect to } |\cdot|^M_\omega, \text{ where}\\ |\boldsymbol{\eta}|^M_\omega &:= \left\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\}^{1/2}.\end{aligned}$$

Let

$$\begin{aligned}a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),\\ B_M(\boldsymbol{\zeta},\boldsymbol{\eta}) &:= \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\zeta},\boldsymbol{\eta} \in \mathbf{V}(\omega),\\ L(\boldsymbol{\eta}) &:= \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\eta} \in \mathbf{V}(\omega),\end{aligned}$$

where the functions $\varphi^{\alpha\beta} \in L^2(\omega)$ are given, and let $B^\sharp_M$ and $L^\sharp_M$ denote the unique continuous extensions from $\mathbf{V}(\omega)$ to $\mathbf{V}^\sharp_M(\omega)$ of the bilinear form B_M and linear form L.

Then there is exactly one solution to the associated two-dimensional variational problem $\mathcal{P}^\sharp_M(\omega)$ of Theorem 9.1:
Find $\boldsymbol{\zeta}$ such that

$$\boldsymbol{\zeta} \in \mathbf{V}^\sharp_M(\omega) \text{ and } B^\sharp_M(\boldsymbol{\zeta},\boldsymbol{\eta}) = L^\sharp_M(\boldsymbol{\eta})$$

for all $\boldsymbol{\eta} \in \mathbf{V}^\sharp_M(\omega)$.

Proof. The assumption $\mathbf{V}_0(\omega) = \{\mathbf{0}\}$ means that the seminorm $|\cdot|^M_\omega$ is a norm over the space $\mathbf{V}(\omega)$. The linear form $L : \mathbf{V}(\omega) \to \mathbb{R}$ and the bilinear form $B_M : \mathbf{V}(\omega) \times \mathbf{V}(\omega) \to \mathbb{R}$ are clearly continuous with respect to this norm. Besides,

$$B_M(\boldsymbol{\eta},\boldsymbol{\eta}) \geq c_e^{-1}\sqrt{a_0}(|\boldsymbol{\eta}|^M_\omega)^2 \text{ for all } \boldsymbol{\eta} \in \mathbf{V}(\omega),$$

since there exist constants c_e and a_0 such that

$$\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ and $a(y) \geq a_0 > 0$ for all $y \in \overline{\omega}$. These properties remain valid on the space $\mathbf{V}_M^\sharp(\omega)$ since $\mathbf{V}(\omega)$ is by construction dense in $\mathbf{V}_M^\sharp(\omega)$, again with respect to $|\cdot|_\omega^M$. The conclusion thus follows from the Lax–Milgram lemma. $\square$

In order to get physically meaningful formulas, it remains to 'de-scale' the unknown $\boldsymbol{\zeta}$ that satisfies the limit 'scaled' problem $\mathcal{P}_M^\sharp(\omega)$ found in Theorem 9.1. In view of the scaling $\boldsymbol{u}(\varepsilon)(x) = \boldsymbol{u}^\varepsilon(x^\varepsilon)$ for all $x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon$ made on the displacement field (Section 7), we are naturally led to defining for each $\varepsilon > 0$ the 'limit' vector field $\boldsymbol{\zeta}^\varepsilon$ by letting

$$\boldsymbol{\zeta}^\varepsilon := \boldsymbol{\zeta}.$$

Recall that λ^ε and μ^ε denote for each $\varepsilon > 0$ the *actual* Lamé constants of the elastic material constituting the shell. We then have the following immediate corollary to Theorems 9.1 and 9.2; naturally, the existence and uniqueness results of Theorem 9.2 apply *verbatim* to the de-scaled problem $\mathcal{P}_M^{\sharp\varepsilon}(\omega)$ (for this reason, they are not reproduced here).

Theorem 9.3: The two-dimensional equations of a linearly elastic 'generalized membrane' shell. Let the assumptions and definitions not repeated here be as in Theorems 9.1 and 9.2. Let

$$\begin{aligned}
a^{\alpha\beta\sigma\tau,\varepsilon} &:= \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),\\
B_M^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta}) &:= \varepsilon\int_\omega a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\zeta},\boldsymbol{\eta} \in \mathbf{V}(\omega),\\
L_M^\varepsilon(\boldsymbol{\eta}) &:= \int_\omega \varphi^{\alpha\beta,\varepsilon}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \text{ for } \boldsymbol{\eta} \in \mathbf{V}(\omega),\\
\varphi^{\alpha\beta,\varepsilon} &:= \varepsilon\varphi^{\alpha\beta},
\end{aligned}$$

and let $B_M^{\sharp\varepsilon}$ and $L_M^{\sharp\varepsilon}$ denote the unique continuous extensions from $\mathbf{V}(\omega)$ to $\mathbf{V}_M^\sharp(\omega)$ of the bilinear form B_M^ε and linear form L_M^ε. Then the limit vector field $\boldsymbol{\zeta}^\varepsilon$ satisfies the following two-dimensional variational problem $\mathcal{P}_M^{\sharp\varepsilon}(\omega)$ of a linearly elastic generalized membrane shell:

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_M^\sharp(\omega) \text{ and } B_M^{\sharp\varepsilon}(\boldsymbol{\zeta}^\varepsilon,\boldsymbol{\eta}) = L_M^{\sharp\varepsilon}(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_M^\sharp(\omega).$$

Equivalently, the field $\boldsymbol{\zeta}^\varepsilon$ satisfies the following minimization problem

$$\begin{aligned}
&\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_M^\sharp(\omega) \text{ and } j_M^{\sharp\varepsilon}(\boldsymbol{\zeta}^\varepsilon) = \inf_{\boldsymbol{\eta}\in\mathbf{V}_M^\sharp(\omega)} j_M^{\sharp\varepsilon}(\boldsymbol{\eta}), \text{ where}\\
&j_M^{\sharp\varepsilon}(\boldsymbol{\eta}) := \frac{1}{2}B_M^{\sharp\varepsilon}(\boldsymbol{\eta},\boldsymbol{\eta}) - L_M^{\sharp\varepsilon}(\boldsymbol{\eta}).
\end{aligned}$$

$\square$

Each one of the two formulations found in Theorem 9.3 constitutes the two-dimensional equations of a linearly elastic generalized membrane shell. The functional $j_M^{\sharp\varepsilon} : \mathbf{V}_M^\sharp(\omega) \to \mathbb{R}$ is the *two-dimensional energy* and the functional

$$\boldsymbol{\eta} \in \mathbf{V}_M^\sharp(\omega) \to \frac{1}{2} B_M^{\sharp\varepsilon}(\boldsymbol{\eta}, \boldsymbol{\eta})$$

is the *two-dimensional strain energy* of a *linearly elastic generalized membrane shell*. The functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the *contravariant components* of the *two-dimensional elasticity tensor of the shell*, already encountered in the two-dimensional equations of a linearly elastic elliptic membrane shell (Theorem 8.3).

The bilinear form $B_M^{\sharp\varepsilon}$ found in the variational equations of a linearly elastic *generalized membrane* shell is an extension of the bilinear form B_M^ε already found in the variational equations of a linearly elastic *elliptic membrane* shell (Theorem 8.3). Recall that both kinds constitute together the *linearly elastic membrane shells.*

Under the essential assumptions that the space $\mathbf{V}_F(\omega)$ reduces to $\{\mathbf{0}\}$ and that the forces are admissible, we have therefore justified by a convergence result (Theorem 9.1) the two-dimensional equations of a linearly elastic generalized membrane shell. In so doing, we have also justified the formal asymptotic approach of Caillerie and Sanchez-Palencia (1995*b*) when 'bending is badly inhibited', according to the terminology of E. Sanchez-Palencia.

The asymptotic analysis of Ciarlet and Lods (1996*d*) described in this section has been extended by Şlicaru (1998) to linearly elastic shells whose middle surface 'has no boundary', such as a torus.

Among linearly elastic shells, generalized membrane shells possess distinctive characteristics that set them apart.

While *forces* applied to a family of elliptic membrane or flexural shells are not subjected to any restriction (see Sections 8 and 10), body forces applied to a family of generalized membrane shells can no longer be accounted for by an arbitrary linear form of the form

$$\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \to \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\, \mathrm{d}x^\varepsilon,$$

that is, with *arbitrary* contravariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$. They must be *admissible for the three-dimensional equations*, in order that the associated scaled linear forms be in particular continuous with respect to the norm

$$\boldsymbol{v} \to \left\{ \sum_{i,j} |e_{i\|j}(\varepsilon; \boldsymbol{v})|^2_{0,\Omega} \right\}^{1/2}$$

and uniformly so with respect to $\varepsilon > 0$ (Section 9.2).

The linear form found in the variational equations of the limit two-dimensional problem for such a shell is likewise subjected to a restriction. On the dense subspace $\mathbf{V}(\omega)$ of the space $\mathbf{V}_M^\sharp(\omega)$, it must be of the form

$$\boldsymbol{\eta} \to \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y$$

(Theorem 9.1). In other words, the applied forces must also be *admissible for the two-dimensional equations*, in such a way that the linear form appearing therein must be an element of the *dual space* of $\mathbf{V}_M^\sharp(\omega)$.

As this dual space may be quite 'small', the limit variational problem, which otherwise satisfies all the assumptions of the Lax–Milgram lemma (Theorem 9.2), possesses the unusual feature that its solution may no longer exist if the data undergo arbitrarily small, yet arbitrarily smooth, perturbations! Another unusual feature of this problem is that the space $\mathbf{V}_M^\sharp(\omega)$ in which its solution is sought may not necessarily be a space of distributions!

Such variational problems fall in the category of 'sensitive problems' introduced by Lions and Sanchez-Palencia (1994). Since then, such problems have been extensively studied. See, in particular, Lions and Sanchez-Palencia (1996, 1997*a*,*b*, 1998, 2000), Pitkäranta and Sanchez-Palencia (1997), Sanchez-Palencia (1999, 2000), Leguillon, Sanchez-Hubert and Sanchez-Palencia (1999), Delfour (1999).

Examples of linearly elastic generalized membrane shells are numerous and, in this respect, those given in Section 9.1 constitute only a small sample. In each case, however, the proof that the space $\mathbf{V}_F(\omega)$ reduces to $\{\mathbf{0}\}$, the identification of the corresponding space $\mathbf{V}_M^\sharp(\omega)$, and the identification of 'admissible' applied forces usually require delicate analyses. In this respect, see notably Sanchez-Hubert and Sanchez-Palencia (1997, Chapter 7, Sections 2 and 4), Lions and Sanchez-Palencia (1997*b*, 1998), Karamian (1998*b*), Lods and Mardare (1998*a*), Mardare (1998*c*), Gérard and Sanchez-Palencia (2000).

The occurrence of boundary layers in generalized membrane shells is studied in Karamian, Sanchez-Hubert and Sanchez-Palencia (2000).

10. 'Flexural' shells

A shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, subjected to a boundary condition of place along a portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$, where $\gamma_0 \subset \gamma$, as its middle curve, is called a *linearly elastic 'flexural' shell* if its associated space

$$\mathbf{V}_F(\omega) = \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\}$$

contains *nonzero* functions.

The purpose of this section is to identify and to mathematically justify the *two-dimensional equations of a linearly elastic flexural shell*, by showing how the convergence of the three-dimensional displacements can be established in *ad hoc* function spaces as the thickness of such a shell approaches zero.

10.1. Definition and examples

Let ω be a domain in $\mathbb{R}^2$ with boundary γ and let $\boldsymbol{\theta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\partial_1\boldsymbol{\theta}(y), \partial_2\boldsymbol{\theta}(y)$ are linearly independent at every point $y \in \overline{\omega}$. A shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is called a *linearly elastic 'flexural' shell* if the following two conditions are simultaneously satisfied.

(i) The shell is subjected to a (homogeneous) *boundary condition of place* along a portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$ as its middle curve (*i.e.*, the displacement vanishes on this portion), where the subset $\gamma_0 \subset \gamma$ satisfies

$$\text{length } \gamma_0 > 0.$$

(ii) Define the space

$$\begin{aligned}\mathbf{V}_F(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) :& \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega&\big\}\end{aligned}$$

(∂_ν denoting the outer normal derivative operator along γ). Then the space $\mathbf{V}_F(\omega)$ contains nonzero functions; equivalently,

$$\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}.$$

We recall that the functions

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$$

denote the covariant components of the *linearized change of metric tensor* associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the surface S.

In other words, there exist nonzero admissible linearized inextensional displacements $\eta_i\boldsymbol{a}^i$ of the middle surface S. 'Admissible' means that they satisfy *two-dimensional boundary conditions of clamping* along the curve $\boldsymbol{\theta}(\gamma_0)$, expressed here by means of the boundary conditions $\eta_i = \partial_\nu\eta_3$ on γ_0 on the associated field $\boldsymbol{\eta} = (\eta_i)$ (these boundary conditions will be interpreted later). 'Linearized inextensional' indicates that the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ are the *linearizations* with respect to $\boldsymbol{\eta} = (\eta_i)$ of the covariant components of the *exact* change of metric tensor associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the surface S; *cf.* Section 4.

A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a cylinder and which is subjected to a boundary condition of place (*i.e.*, of vanishing displacement field) along a portion (solid black in the figure) of its lateral face whose

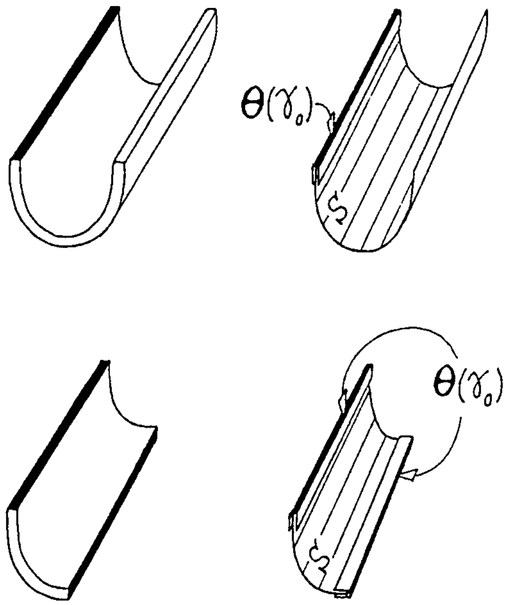

Fig. 10.1. Linearly elastic 'flexural' shells

middle curve $\boldsymbol{\theta}(\gamma_0)$ is contained in one or two generatrices of S provides an instance of a linearly elastic *flexural* shell, that is, one for which the associated space $\mathbf{V}_F(\omega)$ contains nonzero functions $\boldsymbol{\eta}$; see Figure 10.1. The *two-dimensional boundary conditions of clamping* $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0 that will eventually be inherited by the limit two-dimensional equations are so named because they mean that the points of, and the tangent spaces to, the deformed and undeformed middle surfaces coincide along the set $\boldsymbol{\theta}(\gamma_0)$, as suggested in the 'two-dimensional' figures.

A shell whose middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ is a portion of a cone excluding its vertex and which is subjected to a boundary condition of place along a portion (solid black in the figure) of its lateral face whose middle curve $\boldsymbol{\theta}(\gamma_0)$ is contained in one generatrix of S provides another example of a linearly elastic *flexural* shell, since again $\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}$ in this case. See Figure 10.2, where the two-dimensional boundary conditions of clamping inherited by the limit two-dimensional equations are again suggested in the 'two-dimensional' figure.

Incidentally, a comparison with the cylindrical and conical shells shown in Figure 9.2 illustrates the crucial role played by the set $\boldsymbol{\theta}(\gamma_0)$ in determining the type of shell!

A plate, subjected to a boundary condition of place along any portion (solid black in the figure) of its lateral face whose middle line γ_0 satisfies

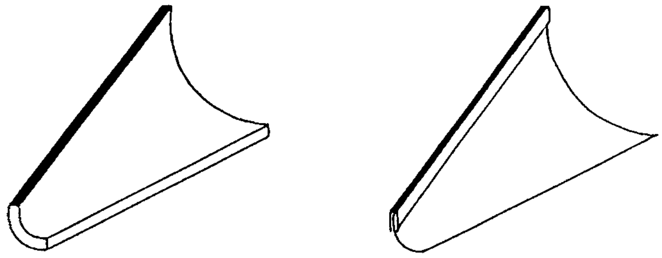

Fig. 10.2. Another example of a linearly elastic 'flexural' shell

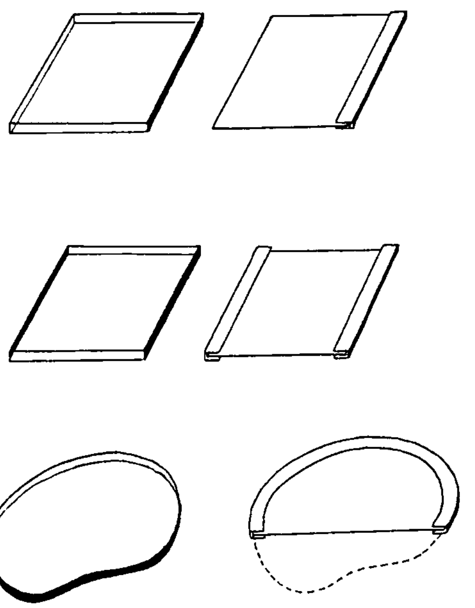

Fig. 10.3. Another example of a linearly elastic 'flexural' shell: a plate

length $\gamma_0 > 0$, provides an instance of a linearly elastic *flexural* shell since

$$\mathbf{V}_F(\omega) \supset \{\boldsymbol{\eta} = (0, 0, \eta_3) : \eta_3 \in H_0^2(\omega)\} \neq \{\mathbf{0}\}$$

in each case. See Figure 10.3, where the two-dimensional boundary conditions of clamping inherited by the limit two-dimensional equations are again suggested in the 'two-dimensional' figures.

The definition of a linearly elastic flexural shell thus depends only on the subset of the lateral face where the shell is subjected to a boundary condition of place (via the set γ_0) and on the geometry of the middle surface of the shell.

10.2. Convergence of the scaled displacements as the thickness approaches zero

We now establish the main results of this section. Consider a family of linearly elastic flexural shells with thickness $2\varepsilon > 0$, with each having the same middle surface $S = \boldsymbol{\theta}(\omega)$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, the assumptions on the data being as in Theorem 10.1 below.

Then the solutions $\boldsymbol{u}(\varepsilon)$ of the associated scaled three-dimensional problems $\mathcal{P}(\varepsilon;\Omega)$ (Theorem 7.1) converge in $\mathbf{H}^1(\Omega)$ as $\varepsilon \to 0$ toward a limit $\boldsymbol{u}$ and this limit, which is independent of the transverse variable x_3, can be identified with the solution $\overline{\boldsymbol{u}}$ of a two-dimensional variational problem $\mathcal{P}_F(\omega)$ posed over the set ω.

The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ appearing in the next theorem respectively represent the covariant components of the *linearized change of metric* and *linearized change of curvature tensors* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S.

Note that the assumption on the applied body forces made in the next theorem corresponds to letting $a = 2$ in Theorem 7.1. That $a = 2$ is indeed the 'correct' exponent in this case can be justified in two different ways.

It is easily checked that this choice is the only one that lets the applied body forces enter (via the functions p^i) the right-hand sides of the variational equations in the 'limit' variational problem $\mathcal{P}_F(\omega)$ satisfied by $\overline{\boldsymbol{u}}$.

Otherwise, the number a can be considered *a priori* as an unknown. Then a *formal* (but careful!) asymptotic analysis of the scaled unknown $\boldsymbol{u}(\varepsilon)$ shows that, for a family of linearly elastic flexural shells, the exponent a must be set equal to 2, again in order that the applied body forces contribute to the 'limit' variational problem; *cf.* Miara and Sanchez-Palencia (1996).

The next result is due to Ciarlet, Lods and Miara (1996, Theorem 5.1); a complete proof is also given in Ciarlet (2000, Theorem 6.2-1).

Theorem 10.1: Convergence of the scaled displacements. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^3)$. Consider a family of linearly elastic flexural shells with thickness 2ε approaching zero, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve. Further, assume that there exist constants $\lambda > 0$ and $\mu > 0$ and functions

$f^i \in L^2(\Omega)$ independent of ε such that

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

(the notation is that of Section 7).

Let $\boldsymbol{u}(\varepsilon)$ denote, for sufficiently small $\varepsilon > 0$, the solution of the associated scaled three-dimensional problem $\mathcal{P}(\varepsilon;\Omega)$ (Theorem 7.1). Then there exists $\boldsymbol{u} \in \mathbf{H}^1(\Omega)$ satisfying $\boldsymbol{u} = \mathbf{0}$ on $\Gamma_0 = \gamma_0 \times [-1,1]$ such that

$$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \text{ in } \mathbf{H}^1(\Omega) \text{ as } \varepsilon \to 0,$$
$$\text{where } \boldsymbol{u} = (u_i) \text{ is independent of the transverse variable } x_3.$$

Furthermore, the average $\overline{\boldsymbol{u}} := \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, \mathrm{d}x_3$ satisfies the following two-dimensional variational problem $\mathcal{P}_F(\omega)$:

$$\begin{aligned}
&\overline{\boldsymbol{u}} = (\overline{u}_i) \in \mathbf{V}_F(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\
&\qquad\qquad\qquad \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\}, \\
&\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\overline{\boldsymbol{u}}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y = \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y
\end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$. Here

$$\begin{aligned}
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \\
\rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) \\
&\quad + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau, \\
a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \\
p^i &:= \int_{-1}^{1} f^i\, \mathrm{d}x_3.
\end{aligned}$$

Sketch of proof. **(i)** The proof rests on the same crucial three-dimensional inequality of Korn's type that was already needed for the asymptotic analysis of 'generalized membrane' shells (Theorem 9.1). For a family of linearly elastic shells, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, there exists a constant C such that, for sufficiently small $\varepsilon > 0$,

$$\|\boldsymbol{v}\|_{1,\Omega} \le \frac{C}{\varepsilon}\bigg\{\sum_{i,j} |e_{i\|j}(\varepsilon;\boldsymbol{v})|^2_{0,\Omega}\bigg\}^{1/2}$$

for all $\boldsymbol{v} \in \mathbf{V}(\Omega)$, where

$$\mathbf{V}(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{v} = \mathbf{0} \text{ on } \gamma_0 \times [-1,1]\},$$

and the functions $e_{i\|j}(\varepsilon; \boldsymbol{v})$ are the scaled linearized strains appearing in Theorem 7.1.

(ii) Letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of problem $\mathcal{P}(\varepsilon; \Omega)$ (Theorem 7.1) and using the three-dimensional inequality of Korn's type used in (i), we obtain a chain of inequalities showing that the norms $\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}$ and $|\frac{1}{\varepsilon} e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon))|_{0,\Omega}$ are bounded independently of ε.

Thus there exists a subsequence, still denoted by $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$ for notational convenience, such that

$$\begin{aligned} \boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} \text{ in } \mathbf{H}^1(\Omega), \text{ and thus } \boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \text{ in } \mathbf{L}^2(\Omega), \\ \frac{1}{\varepsilon} e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon)) \rightharpoonup e^1_{i\|j} \text{ in } L^2(\Omega). \end{aligned}$$

(iii) The above convergence, combined with the asymptotic behaviour of the functions $\Gamma^p_{ij}(\varepsilon)$, $A^{ijkl}(\varepsilon)$, and $g(\varepsilon)$, then implies that the vector field $\boldsymbol{u}$ is independent of x_3. Further, the average $\overline{\boldsymbol{u}} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}\, dx_3$ belongs to the space $\mathbf{V}_F(\omega)$, and the field $\boldsymbol{u}$ and the functions $e^1_{i\|j}$ are related by

$$\begin{aligned} -\partial_3 e^1_{\alpha\|\beta} &= \rho_{\alpha\beta}(\boldsymbol{u}), \\ e^1_{\alpha\|3} &= 0, \quad e^1_{3\|3} = -\frac{\lambda}{\lambda + 2\mu} a^{\alpha\beta} e^1_{\alpha\|\beta}. \end{aligned}$$

(iv) In the variational equations of problem $\mathcal{P}(\varepsilon; \Omega)$, let $\boldsymbol{v} = (v_i(\varepsilon))$, where the functions $v_i(\varepsilon)$ are of the form

$$v_\alpha(\varepsilon) = \eta_\alpha - \varepsilon x_3(\partial_\alpha \eta_3 + 2 b^\sigma_\alpha \eta_\sigma) \text{ and } v_3(\varepsilon) = \eta_3$$

for some *fixed* $\boldsymbol{\eta} = (\eta_i)$ in the space $\mathbf{V}_F(\omega)$, and let ε approach zero. Then the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon)$ and $g(\varepsilon)$, combined with the relations found in (iii), shows that the average $\overline{\boldsymbol{u}} \in \mathbf{V}_F(\omega)$ indeed satisfies the variational equations of the two-dimensional problem $\mathcal{P}_F(\omega)$ stated in the statement of the theorem.

The solution to $\mathcal{P}_F(\omega)$ being *unique*, the convergence $\boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u}$ in $\mathbf{H}^1(\Omega)$ and $\boldsymbol{u}(\varepsilon) \to \boldsymbol{u}$ in $L^2(\Omega)$ established in (ii) for a subsequence thus holds for the *whole* family $(\boldsymbol{u}(\varepsilon))_{\varepsilon>0}$.

(v) Again letting $\boldsymbol{v} = \boldsymbol{u}(\varepsilon)$ in the variational equations of $\mathcal{P}(\varepsilon; \Omega)$ and using the results obtained in (ii)–(iv), we obtain the strong convergence

$$\frac{1}{\varepsilon} e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon)) \to e^1_{i\|j} \text{ in } L^2(\Omega),$$

which in turn implies that

$$\boldsymbol{u}(\varepsilon) \to \boldsymbol{u} \text{ in } \mathbf{H}^1(\Omega),$$

as was to be proved. □

10.3. The two-dimensional equations of a linearly elastic 'flexural' shell

The next theorem recapitulates the definition and assembles the main features of the 'limit' two-dimensional variational problem $\mathcal{P}_F(\omega)$ found at the outcome of the asymptotic analysis carried out in Theorem 10.1.

Theorem 10.2: Existence and uniqueness of solutions. Let ω be a domain in $\mathbb{R}^2$, let γ_0 be a subset of the boundary of ω with length $\gamma_0 > 0$, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$ and such that

$$\begin{aligned}\mathbf{V}_F(\omega) := \big\{&\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ &\eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\big\} \neq \{\mathbf{0}\},\end{aligned}$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3.$$

The associated two-dimensional variational problem $\mathcal{P}_F(\omega)$ found in Theorem 10.1 is as follows. Given $p^i \in L^2(\omega)$, find $\boldsymbol{\zeta} = (\zeta_i)$ satisfying

$$\begin{aligned}&\boldsymbol{\zeta} \in \mathbf{V}_F(\omega), \\ &\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y\end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$, where

$$\begin{aligned}\rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) \\ &\quad + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau, \\ a^{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu\big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\big).\end{aligned}$$

This problem has exactly one solution, which is also the unique solution of the minimization problem:
Find $\boldsymbol{\zeta}$ such that

$$\begin{aligned}&\boldsymbol{\zeta} \in \mathbf{V}_F(\omega) \text{ and } j_F(\boldsymbol{\zeta}) = \inf_{\boldsymbol{\eta}\in\mathbf{V}_F(\omega)} j_F(\boldsymbol{\eta}), \text{ where} \\ &j_F(\boldsymbol{\eta}) := \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y - \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y.\end{aligned}$$

Proof. The existence and uniqueness of a solution to the variational problem $\mathcal{P}_F(\omega)$, or to its equivalent minimization problem, is a consequence of the inequality of Korn's type on a general surface (Theorem 4.4), of the existence of constants c_e and a_0 such that

$$\sum_{\alpha,\beta}|t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$ and $a(y) \geq a_0 > 0$ for all $y \in \overline{\omega}$, and of the Lax–Milgram lemma. $\square$

The minimization problem encountered in Theorem 10.2 (or that in Theorem 10.3 below in its 'de-scaled' formulation) provides an interesting example of a minimization problem with 'equality constraints', namely the relations

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega$$

to be satisfied by the elements $\boldsymbol{\eta}$ of the space $\mathbf{V}_F(\omega)$ over which the functional is to be minimized.

In order to get physically meaningful formulas, we must 'de-scale' the unknowns ζ_i that satisfy the limit 'scaled' problem $\mathcal{P}_F(\omega)$ found in Theorem 10.2. In view of the scalings $u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon)$ for all $x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon$ made on the covariant components of the displacement field (Section 7), we are naturally led to defining, for each $\varepsilon > 0$, the covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the 'limit displacement field' $\zeta_i^\varepsilon \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface S of the shell by letting

$$\zeta_i^\varepsilon := \zeta_i$$

(the vectors $\boldsymbol{a}^i$ forming the contravariant basis at each point of S).

Like those found in the analysis of linearly elastic elliptic membrane shells (Section 8), the fields $\boldsymbol{\zeta}^\varepsilon := (\zeta_i^\varepsilon)$ and $\zeta_i^\varepsilon \boldsymbol{a}^i$ must be carefully distinguished! The former is essentially a convenient mathematical 'intermediary', but only the latter has physical significance.

Recall that $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ represent the contravariant components of the applied body forces *actually* acting on the shell and that λ^ε and μ^ε denote the *actual* Lamé constants of its constituent material. We then have the following immediate corollary to Theorems 10.1 and 10.2; naturally, the existence and uniqueness results of Theorem 10.2 apply *verbatim* to the solution of the 'de-scaled' problem $\mathcal{P}_F^\varepsilon(\omega)$ found in the next theorem (for this reason, they are not reproduced here).

Theorem 10.3: The two-dimensional equations of a linearly elastic 'flexural' shell. Let the assumptions on the data and the definitions of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ be as in Theorem 10.2. Then the vector field $\boldsymbol{\zeta}^\varepsilon := (\zeta_i^\varepsilon)$ formed by the covariant components of the limit displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i$ of the middle surface S satisfies the following two-dimensional variational problem $\mathcal{P}_F^\varepsilon(\omega)$ of a linearly elastic flexural shell:

$$\begin{aligned}\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) :\\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0, \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\},\end{aligned}$$

$$\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i\sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$, where

$$a^{\alpha\beta\sigma\tau,\varepsilon} := \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta} a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),$$

$$p^{i,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\,\mathrm{d}x_3^\varepsilon.$$

Equivalently, the field $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$ satisfies the minimization problem

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_F(\omega) \text{ and } j_F^\varepsilon(\boldsymbol{\zeta}^\varepsilon) = \inf_{\boldsymbol{\eta} \in \mathbf{V}_F(\omega)} j_F^\varepsilon(\boldsymbol{\eta}), \text{ where}$$

$$j_F^\varepsilon(\boldsymbol{\eta}) := \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\,\mathrm{d}y - \int_\omega p^{i,\varepsilon}\eta_i \sqrt{a}\,\mathrm{d}y. \qquad \square$$

Each one of the two formulations found in Theorem 10.3 constitutes the two-dimensional equations of a linearly elastic flexural shell.

We recall that the condition $\mathbf{V}_F(\omega) \neq \{\mathbf{0}\}$, which is the basis of the definition of a linearly elastic flexural shell, means that there exist nonzero ‘admissible linearized inextensional displacements’ of the middle surface, since the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ used in the definition of $\mathbf{V}_F(\omega)$ are the covariant components of the *linearized change of metric tensor* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S; ‘admissible’ means that the fields $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_F(\omega)$ must also satisfy the boundary conditions $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0.

In order to interpret these boundary conditions, let $\eta_i \boldsymbol{a}^i$ be a displacement field of the middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ with smooth enough, but otherwise arbitrary, covariant components $\eta_i : \overline{\omega} \to \mathbb{R}$. The tangent plane at an arbitrary point $\boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y), y \in \overline{\omega}$, of the deformed surface $(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(\overline{\omega})$ is thus spanned by the vectors

$$\partial_\alpha(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(y) = \boldsymbol{a}_\alpha(y) + \partial_\alpha \eta_i(y)\boldsymbol{a}^i(y) + \eta_i(y)\partial_\alpha \boldsymbol{a}^i(y),$$

if these are linearly independent. Since

$$\eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0 \;\Rightarrow\; \eta_i = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0,$$

it follows that

$$\begin{aligned} \boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y) &= \boldsymbol{\theta}(y) \text{ for all } y \in \gamma_0, \\ \partial_\alpha(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)(y) &= \boldsymbol{a}_\alpha(y) + \partial_\alpha \eta_\beta(y)\boldsymbol{a}^\beta(y) \text{ for all } y \in \gamma_0. \end{aligned}$$

These relations thus show that the points of the deformed and undeformed middle surfaces, and their tangent spaces at those points where the vectors $\partial_\alpha(\boldsymbol{\theta} + \eta_i \boldsymbol{a}^i)$ are linearly independent, coincide along the set $\boldsymbol{\theta}(\gamma_0)$. Such ‘two-dimensional boundary conditions of clamping’ are suggested in Figures 10.1 to 10.3.

The functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the *covariant components* of the *linearized change of curvature tensor* associated with a displacement field $\eta_i \boldsymbol{a}^i$ of the middle surface S and the functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the *contravariant components* of the *two-dimensional elasticity tensor of the shell*, already encountered in the two-dimensional equations of linearly elastic elliptic membrane and generalized membrane shells (Theorems 8.3 and 9.3).

Finally, the functional $j_F^\varepsilon : \mathbf{V}_F(\omega) \to \mathbb{R}$ is the *two-dimensional energy* and the functional

$$\boldsymbol{\eta} \in \mathbf{V}_F(\omega) \to \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y$$

is the *two-dimensional strain energy* of a *linearly elastic flexural shell.*

Under the essential assumptions that the space $\mathbf{V}_F(\omega)$ contains nonzero elements, we have therefore justified by a convergence result (Theorem 10.1) the two-dimensional equations of a linearly elastic flexural shell. In so doing, we have justified the formal asymptotic approach of Sanchez-Palencia (1990) (see also Miara and Sanchez-Palencia (1996) and Caillerie and Sanchez-Palencia (1995*b*)) when 'bending is not inhibited', according to the terminology of E. Sanchez-Palencia.

Due credit should be given in this respect to Sanchez-Palencia (1989*a*) for recognizing the central role played by the space $\mathbf{V}_F(\omega)$ in the classification of linearly elastic shells.

The above convergence analysis also substantiate an important observation. In a *flexural shell*, body forces of order $O(\varepsilon^2)$ produce an $O(1)$ limit displacement field. By contrast, body forces of order $O(1)$ are required to also produce an $O(1)$ limit displacement field in an *elliptic membrane shell*; *cf.* Section 8.

Membrane and flexural shells thus exhibit strikingly different limit behaviour!

After the original work of Ciarlet, Lods and Miara (1996) described in this section, the asymptotic analysis of linearly elastic flexural shells underwent several refinements and generalizations, which include another proof of Theorem 10.1 by means of Γ-convergence theory (Genevey 1999), an asymptotic analysis of linearly elastic flexural shells with variable thickness (Busse 1998) or made with a nonhomogeneous and anisotropic material (Giroud 1998), the convergence of the stresses and the explicit forms of the limit stresses (Collard and Miara 1999), and an asymptotic analysis of the associated eigenvalue problem (Kesavan and Sabu 2000) and time-dependent problem (Xiao Li-Ming 200x*a*).

11. Koiter's equations

Founding his approach on *a priori* assumptions of a *geometrical* and *mechanical* nature about the three-dimensional displacements and stresses when the thickness is 'small', W. T. Koiter proposed in the sixties a two-dimensional shell model that has quickly acquired widespread popularity within the computational mechanics community.

After briefly describing the genesis of these equations, which were encountered in Section 4, we review in this section their main mathematical properties, such as the existence, uniqueness, and regularity of their solution, or their formulation as a boundary value problem. We also show how they can be extended to shells whose middle surface has little regularity and we describe the closely related Budiansky–Sanders equations.

It is remarkable that Koiter's equations can be fully justified for all types of shells, even though it is clear that these equations cannot be recovered as the outcome of an asymptotic analysis of the three-dimensional equations, since Sections 8 to 10 have exhausted all such possible outcomes!

More specifically, we also show in this section that, for each category of linearly elastic shells (elliptic membrane, generalized membrane, or flexural), the solution of Koiter's equation and the average through the thickness of the three-dimensional solution have the same asymptotic behaviour in *ad hoc* function spaces as $\varepsilon \to 0$.

So, even though Koiter's linear model is not a limit model, it is in this sense the 'best' two-dimensional one for linearly elastic shells!

11.1. Genesis; existence, uniqueness, and regularity of solutions; formulation as a boundary value problem

Let ω be a domain in $\mathbb{R}^2$ with boundary γ, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$, and let γ_0 be a portion of γ that satisfies length $\gamma_0 > 0$.

Consider as in the previous sections a linearly elastic shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and thickness $2\varepsilon > 0$, that is, a linearly elastic body whose reference configuration is the set $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$, where

$$\begin{aligned} \Omega^\varepsilon &:= \omega \times]-\varepsilon, \varepsilon[, \\ \boldsymbol{\Theta}(y, x_3^\varepsilon) &:= \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}_3(y) \text{ for all } (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon. \end{aligned}$$

The material constituting the shell is *homogeneous* and *isotropic* and the reference configuration is a *natural state*, so that the material is characterized by its two Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. The shell is subjected to a *boundary condition of place* along the portion $\boldsymbol{\Theta}(\Gamma_0^\varepsilon)$ of its lateral face, where $\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon]$, that is, the three-dimensional displacement vanishes on $\boldsymbol{\Theta}(\Gamma_0^\varepsilon)$. Finally, the shell is subjected to *applied body forces* in its interior $\boldsymbol{\Theta}(\Omega^\varepsilon)$, their densities being given by their contravariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$.

In a seminal work, John (1965, 1971) showed that, if the thickness of such a shell is small enough, the state of stress is 'approximately planar' and the stresses parallel to the middle surface vary 'approximately linearly' across the thickness, at least 'away from the lateral face'. In Koiter's approach (Koiter 1960, 1966, 1970), these approximations are taken as an *a priori* assumption of a *mechanical* nature and combined with another *a priori* assumption of a *geometrical* nature, called the Kirchhoff–Love assumption: any point on a normal to the middle surface remains on the normal to the deformed middle surface after the deformation has taken place and the distance between such a point and the middle surface remains constant. In fact, this assumption is required to hold only 'to within the first order' in the linearized theory considered in this section.

Taking these two *a priori* assumptions into account, Koiter then shows that the displacement field across the thickness of the shell can be completely determined from the sole knowledge of the displacement field of the middle surface S, and he identifies the two-dimensional problem, that is, posed over the two-dimensional set $\overline{\omega}$, that this displacement field should satisfy. As in the two-dimensional theories encountered so far, the unknown is a vector field, now denoted by $\boldsymbol{\zeta}_K^\varepsilon = (\zeta_{i,K}^\varepsilon) : \overline{\omega} \to \mathbb{R}^3$, whose components $\zeta_{i,K}^\varepsilon : \overline{\omega} \to \mathbb{R}$ are the covariant components of the displacement field of the middle surface S. This means that $\zeta_{i,K}^\varepsilon(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\theta}(y)$; see Figure 11.1.

In their linearized version, the equations found by Koiter consist in solving the following variational problem $\mathcal{P}_K^\varepsilon(\omega)$:
Find $\boldsymbol{\zeta}_K^\varepsilon = (\zeta_{K,i}^\varepsilon)$ such that

$$\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon)\rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\,\mathrm{d}y \\ = \int_\omega p^{i,\varepsilon}\eta_i \sqrt{a}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$, where

$$\begin{aligned} a^{\alpha\beta\sigma\tau,\varepsilon} &:= \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \\ \gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \\ \rho_{\alpha\beta}(\boldsymbol{\eta}) &:= \partial_{\alpha\beta}\eta_3 - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3 - b^\sigma_\alpha b_{\sigma\beta}\eta_3 \\ &\quad + b^\sigma_\alpha(\partial_\beta\eta_\sigma - \Gamma^\tau_{\beta\sigma}\eta_\tau) + b^\tau_\beta(\partial_\alpha\eta_\tau - \Gamma^\sigma_{\alpha\tau}\eta_\sigma) \\ &\quad + (\partial_\alpha b^\tau_\beta + \Gamma^\tau_{\alpha\sigma}b^\sigma_\beta - \Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\eta_\tau, \\ p^{i,\varepsilon} &:= \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\,\mathrm{d}x_3^\varepsilon \end{aligned}$$

(the functions $a^{\alpha\beta}, b_{\alpha\beta}, b^\sigma_\alpha, \Gamma^\sigma_{\alpha\beta}$, and a defined as usual: see Section 4).

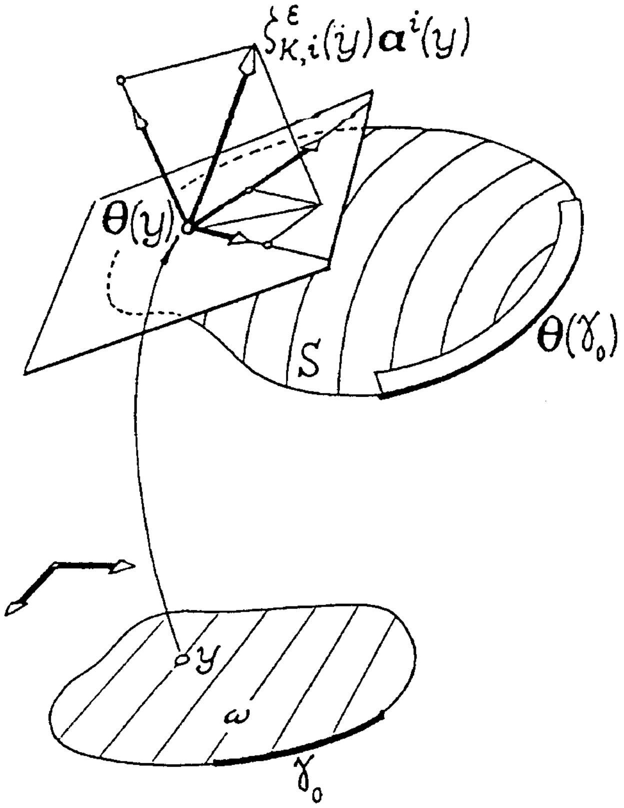

Fig. 11.1. The three unknowns in Koiter's equations are the covariant components $\zeta^{\varepsilon}_{i,K} : \overline{\omega} \to \mathbb{R}$ of the displacement field $\zeta^{\varepsilon}_{i,K}\boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the middle surface S; this means that, for each $y \in \overline{\omega}$, $\zeta^{\varepsilon}_{i,K}(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\theta}(y) \in S$

The functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the customary covariant components of the *linearized change of metric* and *linearized change of curvature tensors* associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the middle surface S and the functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the customary contravariant components of the *two-dimensional elasticity tensor of the shell.*

Note that Destuynder (1985, 1990) has found an illuminating way of deriving the same linear Koiter equations from three-dimensional elasticity, which uses *a priori* assumptions only of a geometrical nature. Note also that the linearized Kirchhoff–Love assumption has been *a posteriori* justified for linearly elastic elliptic membrane shells by Lods and Mardare (1998*b*, 2000*b*).

The existence and uniqueness of a solution to problem $\mathcal{P}_K^\varepsilon(\omega)$, which essentially follow from the $\mathbf{V}_K(\omega)$-ellipticity of the bilinear form, was first established by Bernadou and Ciarlet (1976); a more natural proof was subsequently proposed by Ciarlet and Miara (1992*b*), then combined with the first one in Bernadou, Ciarlet and Miara (1994). The existence and uniqueness of the solution to the time-dependent Koiter equations have recently been established by Xiao Li-Ming (1999).

Theorem 11.1: Existence and uniqueness of solutions. Let ω be a domain in $\mathbb{R}^2$, let γ_0 be a subset of $\gamma = \partial\omega$ with length $\gamma_0 > 0$, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$.

Then the variational problem $\mathcal{P}_K^\varepsilon(\omega)$ has exactly one solution, which is also the unique solution to the minimization problem:
Find $\boldsymbol{\zeta}_K^\varepsilon = (\zeta_{K,i}^\varepsilon)$ such that

$$\boldsymbol{\zeta}_K^\varepsilon \in \mathbf{V}_K(\omega) \text{ and } j_K^\varepsilon(\boldsymbol{\zeta}_K^\varepsilon) = \inf_{\boldsymbol{\eta} \in \mathbf{V}_K(\omega)} j_K^\varepsilon(\boldsymbol{\eta}), \text{ where}$$
$$j_K^\varepsilon(\boldsymbol{\eta}) := \frac{1}{2} \int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, \mathrm{d}y - \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y.$$

Proof. The assumptions $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ imply that $p^{i,\varepsilon} \in L^2(\omega)$. The existence and uniqueness of a solution to the variational problem $\mathcal{P}_K^\varepsilon(\omega)$, or to its equivalent minimization problem, are consequences of the inequality of Korn's type on a general surface (Theorem 4.4), of the existence of a constant c_e such that

$$\sum |t_{\alpha\beta}|^2 \leq c_e a^{\alpha\beta\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$, of the existence of a_0 such that $a(y) \geq a_0 > 0$ for all $y \in \overline{\omega}$, and of the Lax–Milgram lemma. $\square$

We next derive the boundary value problem that is (at least formally) equivalent to Koiter's variational problem $\mathcal{P}_K^\varepsilon(\omega)$. We also state a regularity result that provides instances where the *weak solution* (the solution to the variational problem) becomes a *classical solution* (a solution to the boundary value problem).

Theorem 11.2: Regularity of solutions; formulation as a boundary value problem. (a) Assume that the boundary γ of ω and the functions $p^{i,\varepsilon}$ are sufficiently smooth. Then, if the solution $\boldsymbol{\zeta}_K^\varepsilon = (\zeta_{K,i}^\varepsilon)$ to the variational problem $\mathcal{P}_K^\varepsilon(\omega)$ (Theorem 11.1) is sufficiently smooth, it is also a solution to the following boundary value problem:

$$\begin{aligned}
m^{\alpha\beta,\varepsilon}|_{\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta} m^{\alpha\beta,\varepsilon} - b_{\alpha\beta} n^{\alpha\beta,\varepsilon} &= p^{3,\varepsilon} \text{ in } \omega,\\
-(n^{\alpha\beta,\varepsilon} + b^\alpha_\sigma m^{\sigma\beta,\varepsilon})|_\beta - b^\alpha_\sigma (m^{\sigma\beta,\varepsilon}|_\beta) &= p^{\alpha,\varepsilon} \text{ in } \omega,\\
\zeta^\varepsilon_{i,K} = \partial_\nu \zeta^\varepsilon_{3,K} &= 0 \text{ on } \gamma_0,\\
m^{\alpha\beta,\varepsilon}\nu_\alpha\nu_\beta &= 0 \text{ on } \gamma_1,\\
(m^{\alpha\beta,\varepsilon}|_\alpha)\nu_\beta + \partial_\tau(m^{\alpha\beta,\varepsilon}\nu_\alpha\tau_\beta) &= 0 \text{ on } \gamma_1,\\
(n^{\alpha\beta,\varepsilon} + 2b^\alpha_\sigma m^{\sigma\beta,\varepsilon})\nu_\beta &= 0 \text{ on } \gamma_1,
\end{aligned}$$

where $\gamma_1 := \gamma - \gamma_0, (\nu_\alpha)$ is the unit outer normal vector along γ, $\tau_1 := -\nu_2, \tau_2 := \nu_1$, $\partial_\tau\theta := \tau_\alpha\partial_\alpha\theta$ denotes the tangential derivative of θ in the direction of the vector (τ_α),

$$n^{\alpha\beta,\varepsilon} := \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon_K), \quad m^{\alpha\beta,\varepsilon} := \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon_K),$$

and finally, for an arbitrary tensor with twice differentiable covariant components $n^{\alpha\beta}$,

$$\begin{aligned}
n^{\alpha\beta}|_\beta &:= \partial_\beta n^{\alpha\beta} + \Gamma^\alpha_{\beta\sigma} n^{\beta\sigma} + \Gamma^\beta_{\beta\sigma} n^{\alpha\sigma},\\
n^{\alpha\beta}|_{\alpha\beta} &:= \partial_\alpha(n^{\alpha\beta}|_\beta) + \Gamma^\sigma_{\alpha\sigma}(n^{\alpha\beta}|_\beta).
\end{aligned}$$

(b) Assume that $\gamma = \gamma_0$ and that, for some integer $m \geq 0$ and some real number $q > 1$, γ is of class $\mathcal{C}^{m+4}$, $\boldsymbol{\theta} \in \mathcal{C}^{m+4}(\overline{\omega}; \mathbb{R}^3)$, $p^{\alpha,\varepsilon} \in W^{m+1,q}(\omega)$, and $p^{3,\varepsilon} \in W^{m,q}(\omega)$. Then

$$\boldsymbol{\zeta}^\varepsilon_K = (\zeta^\varepsilon_i) \in W^{m+3,q}(\omega) \times W^{m+3,q}(\omega) \times W^{m+4,q}(\omega).$$

Proof. For brevity, we give the proof of (a) when $\gamma_0 = \gamma$, in which case

$$\mathbf{V}_K(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times H^2_0(\omega),$$

and we omit the exponents 'ε' and the indices 'K' throughout the proof, that is, we let

$$\boldsymbol{\zeta} := \boldsymbol{\zeta}^\varepsilon_K, \quad n^{\alpha\beta} := a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta}), \quad m^{\alpha\beta} := \frac{1}{3} a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta}), \quad p^i := p^{i,\varepsilon}.$$

Assume that the solution $\boldsymbol{\zeta}$ is smooth in the sense that $n^{\alpha\beta} \in H^1(\omega)$ and $m^{\alpha\beta} \in H^2(\omega)$.

We have already seen in the proof of Theorem 8.2 that

$$\int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y = -\int_\omega \sqrt{a}\big\{(n^{\alpha\beta}|_\beta)\eta_\alpha + b_{\alpha\beta}n^{\alpha\beta}\eta_3\big\}\,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega) \times H^1_0(\omega) \times L^2(\omega)$, hence *a fortiori* for all $\boldsymbol{\eta} \in H^1_0(\omega) \times H^1_0(\omega) \times H^2_0(\omega)$. It thus remains to transform the other integral appearing on the left-hand side of the variational equations, that is,

$$
\begin{aligned}
\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y &= \int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \\
&= \int_\omega \sqrt{a}\,m^{\alpha\beta}\partial_{\alpha\beta}\eta_3\,\mathrm{d}y \\
&+ \int_\omega \sqrt{a}\,m^{\alpha\beta}(2b^\sigma_\alpha\partial_\beta\eta_\sigma - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\eta_3)\,\mathrm{d}y \\
&+ \int_\omega \sqrt{a}\,m^{\alpha\beta}(-2b^\tau_\beta\Gamma^\sigma_{\alpha\tau}\eta_\sigma + b^\sigma_\beta|_\alpha\eta_\sigma - b^\sigma_\alpha b_{\sigma\beta}\eta_3)\,\mathrm{d}y,
\end{aligned}
$$

where $\boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$. Using the symmetry $m^{\alpha\beta} = m^{\beta\alpha}$, the relation $\partial_\beta\sqrt{a} = \sqrt{a}\,\Gamma^\sigma_{\beta\sigma}$, and Green's formula in Sobolev space, we obtain

$$
\begin{aligned}
\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y &= -\int_\omega \sqrt{a}(\partial_\beta m^{\alpha\beta} + \Gamma^\sigma_{\beta\sigma}m^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta}m^{\sigma\beta})\partial_\alpha\eta_3\,\mathrm{d}y \\
&+ 2\int_\omega \sqrt{a}\,m^{\alpha\beta}b^\sigma_\alpha\partial_\beta\eta_\sigma\,\mathrm{d}y \\
&+ \int_\omega \sqrt{a}\,m^{\alpha\beta}(-2b^\tau_\beta\Gamma^\sigma_{\alpha\tau}\eta_\sigma + b^\sigma_\beta|_\alpha\eta_\sigma - b^\sigma_\alpha b_{\sigma\beta}\eta_3)\,\mathrm{d}y.
\end{aligned}
$$

The same Green's formula shows that

$$
\begin{aligned}
-\int_\omega &\sqrt{a}(\partial_\beta m^{\alpha\beta} + \Gamma^\sigma_{\beta\sigma}m^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta}m^{\sigma\beta})\partial_\alpha\eta_3\,\mathrm{d}y \\
&= -\int_\omega \sqrt{a}(m^{\alpha\beta}|_\beta)\partial_\alpha\eta_3\,\mathrm{d}y = \int_\omega \partial_\alpha(\sqrt{a}\,m^{\alpha\beta}|_\beta)\eta_3\,\mathrm{d}y \\
&= \int_\omega \sqrt{a}(m^{\alpha\beta}|_{\alpha\beta})\eta_3\,\mathrm{d}y,
\end{aligned}
$$

$$
2\int_\omega \sqrt{a}\,m^{\alpha\beta}b^\sigma_\alpha\partial_\beta\eta_\sigma\,\mathrm{d}y = -2\int_\omega \sqrt{a}\big\{\partial_\beta(b^\sigma_\alpha m^{\alpha\beta}) + \Gamma^\tau_{\beta\tau}b^\sigma_\alpha m^{\alpha\beta}\big\}\eta_\sigma\,\mathrm{d}y.
$$

Consequently,

$$
\begin{aligned}
\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \;=\; &\int_\omega \sqrt{a}\big\{-2(b^\alpha_\sigma m^{\sigma\beta})|_\beta + (b^\alpha_\beta|_\sigma)m^{\sigma\beta}\big\}\eta_\alpha\,\mathrm{d}y \\
+ &\int_\omega \sqrt{a}\big\{m^{\alpha\beta}|_{\alpha\beta} - b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta}\big\}\eta_3\,\mathrm{d}y.
\end{aligned}
$$

Using in this relation the easily verified formula

$$
(b^\alpha_\sigma m^{\sigma\beta})|_\beta = (b^\alpha_\beta|_\sigma)m^{\sigma\beta} + b^\alpha_\sigma(m^{\sigma\beta}|_\beta)
$$

and the symmetry $b^\alpha_\beta|_\sigma = b^\alpha_\sigma|_\beta$, we finally obtain

$$\int_\omega m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \;=\; -\int_\omega \sqrt{a}\big\{(b^\alpha_\sigma m^{\sigma\beta})|_\beta + b^\alpha_\sigma(m^{\sigma\beta}|_\beta)\big\}\eta_\alpha\,\mathrm{d}y$$
$$-\int_\omega \sqrt{a}\big\{b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta} - m^{\alpha\beta}|_{\alpha\beta}\big\}\eta_3\,\mathrm{d}y.$$

Hence the variational equations

$$\int_\omega \Big\{a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{1}{3}a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}) - p^i\eta_i\Big\}\sqrt{a}\,\mathrm{d}y = 0$$

imply that

$$\int_\omega \sqrt{a}\big\{(n^{\alpha\beta} + b^\alpha_\sigma m^{\sigma\beta})|_\beta + b^\alpha_\sigma(m^{\sigma\beta}|_\beta) + p^\alpha\big\}\eta_\alpha\,\mathrm{d}y$$
$$+\int_\omega \sqrt{a}\big\{b_{\alpha\beta}n^{\alpha\beta} + b^\sigma_\alpha b_{\sigma\beta}m^{\alpha\beta} - m^{\alpha\beta}|_{\alpha\beta} + p^3\big\}\eta_3\,\mathrm{d}y = 0$$

for all $(\eta_i) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$. The stated partial differential equations are thus satisfied in ω.

The regularity result of part (b) is due to Alexandrescu (1994). □

Note that the functions $n^{\alpha\beta}|_\beta$ and $m^{\alpha\beta}|_{\alpha\beta}$ appearing in the boundary value problem are instances of first-order and second-order covariant derivatives of tensor fields, defined here by means of their contravariant components $n^{\alpha\beta}$ or $m^{\alpha\beta}$. The covariant derivatives $n^{\alpha\beta}|_\beta$ also occurred in the boundary value problem associated with a linearly elastic elliptic membrane shell (Theorem 8.2).

Each one of the three formulations found in Theorems 11.1 and 11.2 constitutes the two-dimensional Koiter equations for a linearly elastic shell. We recall that the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the linearized change of metric and change of curvature tensors associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the middle surface S, the functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the contravariant components of the two-dimensional elasticity tensor of the shell. The functions $n^{\alpha\beta,\varepsilon}$ and $m^{\alpha\beta,\varepsilon}$ are the contravariant components of the *stress resultant* and *stress couple*, or *bending moment*, *tensor* fields.

As shown at the end of Section 10, the 'two-dimensional boundary conditions of clamping' $\zeta^\varepsilon_{K,i} = \partial_\nu\zeta^\varepsilon_{K,3} = 0$ on γ_0 state that the points of, and the tangent spaces to, the deformed and undeformed middle surfaces coincide along the set $\boldsymbol{\theta}(\gamma_0)$, as suggested in Figure 11.1.

The functional $j^\varepsilon_K : \mathbf{V}_K(\omega) \to \mathbb{R}$ in Theorem 11.1 is the two-dimensional Koiter energy of a linearly elastic shell. The associated Koiter strain energy,

$$\boldsymbol{\eta} \in \mathbf{V}_K(\omega) \to \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\eta})\gamma_{\alpha\beta}(\boldsymbol{\eta})$$
$$+\frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\eta})\rho_{\alpha\beta}(\boldsymbol{\eta})\Big\}\sqrt{a}\,\mathrm{d}y,$$

is thus the *sum* of the strain energies of a linearly elastic elliptic membrane shell (Section 8) and of a linearly elastic flexural shell (Section 10).

Finally, note that the partial differential equations in ω together with the boundary conditions on γ_1 found in Theorem 11.2 may be viewed as two-dimensional equations of equilibrium, while the equations relating the unknown $\boldsymbol{\zeta}_K^\varepsilon$ and the functions $n^{\alpha\beta,\varepsilon}$ and $m^{\alpha\beta,\varepsilon}$ may be viewed as two-dimensional constitutive equations.

11.2. Justification of Koiter's equations for all types of shells

When it is viewed as a three-dimensional body, the linearly elastic shell described at the beginning of this section is modelled by the variational problem $\mathcal{P}(\Omega^\varepsilon)$ that constituted the point of departure of the asymptotic analyses of Sections 8 to 10. This problem, described in Section 7.1, consists in finding $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ such that

$$\boldsymbol{u}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon) : \boldsymbol{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\},$$

$$\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon} e_{k\|l}^\varepsilon(\boldsymbol{u}^\varepsilon) e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$$

for all $\boldsymbol{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$, where

$$A^{ijkl,\varepsilon} := \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}),$$

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}(\partial_j^\varepsilon v_i^\varepsilon + \partial_i^\varepsilon v_j^\varepsilon) - \Gamma_{ij}^{p,\varepsilon}(\boldsymbol{v}^\varepsilon)$$

(all notation not redefined here is defined in Section 7.1).

The unknown functions u_i^ε in problem $\mathcal{P}(\Omega^\varepsilon)$ represent the covariant components of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the points of the reference configuration $\boldsymbol{\Theta}(\overline{\Omega}^\varepsilon)$; see Figure 7.1.

Now consider a family of such linearly elastic shells, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve. All the linearly elastic shells in such a family are thus either *elliptic membrane*, or *generalized membrane*, or *flexural*, according to the definitions given in Sections 8, 9, and 10. Assume that the assumptions on the data are in each case those that guarantee the convergence of the scaled displacements as the thickness approaches zero (Theorems 8.1, 9.1, and 10.1).

It is then remarkable that, in each case, the asymptotic behaviour as $\varepsilon \to 0$ of the average $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon\,\mathrm{d}x_3^\varepsilon$ of the solution to the three-dimensional variational problem $\mathcal{P}(\Omega^\varepsilon)$ and of the solution $\boldsymbol{\zeta}_K^\varepsilon$ to the two-dimensional Koiter equations formulated as the variational problem $\mathcal{P}_K^\varepsilon(\omega)$ (Theorem 11.1) are identical.

To see this, we proceed as in Ciarlet and Lods (1996*c*, Theorems 2.1 and 2.2) and (1996*d*, Theorems 6.1 and 6.2). We compare the convergence theorems established in Sections 8, 9, and 10 with former results of Destuynder (1985), Sanchez-Palencia (1989*a*, 1989*b*, 1992), and Caillerie and Sanchez-Palencia (1995*a*) (see also Caillerie (1996)) about the asymptotic behaviour of the solution of Koiter's equation as ε approaches zero.

The forthcoming analyses have been recently extended by Xiao Li-Ming (200x *b*), who likewise justified the time-dependent Koiter equations for elliptic membrane and flexural shells.

To begin with, we consider *elliptic membrane shells*, as defined in Section 8.1.

Theorem 11.3: Justification of Koiter's equations for 'elliptic membrane' shells. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Consider a family of linearly elastic elliptic membrane shells, with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and let the assumptions on the data be as in Theorem 8.1 (in particular, $\gamma_0 = \gamma$).

For each $\varepsilon > 0$ let

$$(u_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon) \text{ and } \boldsymbol{\zeta}_K^\varepsilon = (\zeta_{i,K}^\varepsilon) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$$

respectively denote the solutions to the three-dimensional and two-dimensional variational problems $\mathcal{P}(\Omega^\varepsilon)$ and $\mathcal{P}_K^\varepsilon(\omega)$. Also, let

$$\boldsymbol{\zeta} = (\zeta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega)$$

denote the solution to the two-dimensional 'scaled' variational problem $\mathcal{P}_M(\omega)$ (Theorem 8.2), a solution which is thus independent of ε. Then

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \,\mathrm{d}x_3^\varepsilon \to \zeta_\alpha \text{ in } H^1(\omega) \text{ and } \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \,\mathrm{d}x_3^\varepsilon \to \zeta_3 \text{ in } L^2(\omega),$$
$$\zeta_{K,\alpha}^\varepsilon \to \zeta_\alpha \text{ in } H^1(\omega) \text{ and } \zeta_{K,3}^\varepsilon \to \zeta_3 \text{ in } L^2(\omega).$$

Proof. Under the assumptions that there exist constants $\lambda > 0$ and $\mu > 0$ and functions $f^i \in L^2(\Omega)$ independent of ε such that

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

(these are the assumptions on the data for a family of linearly elastic elliptic membrane shells) and that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, then

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon \,\mathrm{d}x_3^\varepsilon = \frac{1}{2}\int_{-1}^{1} u_\alpha(\varepsilon)\,\mathrm{d}x_3 \to \zeta_\alpha \text{ in } H^1(\omega)$$

and

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \,\mathrm{d}x_3^\varepsilon = \frac{1}{2}\int_{-1}^{1} u_3(\varepsilon)\,\mathrm{d}x_3 \to \zeta_3 \text{ in } L^2(\omega)$$

as $\varepsilon \to 0$ are easy corollaries to the fundamental convergence result of Theorem 8.1.

The convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $H^1(\omega) \times H^1(\omega) \times L^2(\omega)$ was first established by Destuynder (1985, Theorem 7.1); it was also noted by Sanchez-Palencia (1989*a*, Theorem 4.1) (see also Caillerie and Sanchez-Palencia (1995*a*)), who observed that it is a consequence of general results in perturbation theory, as found for instance in Sanchez-Palencia (1980). We give here a simple and self-contained proof. Let

$$\begin{aligned}
a^{\alpha\beta\sigma\tau} &:= \frac{2\lambda\mu}{\lambda+2\mu} a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),\\
B_M(\boldsymbol{\zeta},\boldsymbol{\eta}) &:= \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y,\\
B_F(\boldsymbol{\zeta},\boldsymbol{\eta}) &:= \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y,\\
L(\boldsymbol{\eta}) &:= \int_\omega p^i\eta_i\sqrt{a}\,\mathrm{d}y, \text{ where } p^i := \int_{-1}^{1} f^i\,\mathrm{d}x_3\\
\mathbf{V}_M(\omega) &:= H_0^1(\omega)\times H_0^1(\omega)\times L^2(\omega),\\
\|\boldsymbol{\eta}\|_{\mathbf{V}_M(\omega)} &:= \left\{\sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + |\eta_3|_{0,\omega}^2\right\}^{1/2}.
\end{aligned}$$

By virtue of the assumptions on the applied forces, the solution $\boldsymbol{\zeta}_K^\varepsilon$ of the two-dimensional Koiter equations also satisfies the scaled Koiter equations for an elliptic membrane shell, namely,

$$B_M(\boldsymbol{\zeta}_K^\varepsilon,\boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon,\boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta}\in\mathbf{V}_K(\omega).$$

Recall that there exists a constant $c_e > 0$ such that

$$\sum_{\alpha,\beta}|t_{\alpha\beta}|^2 \le c_e a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta}$$

for all $y \in \overline{\omega}$ and all symmetric matrices $(t_{\alpha\beta})$. Hence letting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in these scaled equations and using the inequality of Korn's type on an elliptic surface (Theorem 6.3) shows that the family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{V}_M(\omega)$ and that the families $(\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$.

Consequently, there exists a subsequence, still denoted by $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ for convenience, and there exist $\boldsymbol{\zeta}^* \in \mathbf{V}_M(\omega)$ and $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$ such that

$$\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^* \text{ in } \mathbf{V}_M(\omega) \text{ and } \varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \rightharpoonup \rho_{\alpha\beta}^{-1} \text{ in } L^2(\omega)$$

(as usual weak convergence is denoted by $\rightharpoonup$).

Fix $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$ in the scaled Koiter equations and let $\varepsilon \to 0$; then the above weak convergence yields $B_M(\boldsymbol{\zeta}^*,\boldsymbol{\eta}) = L(\boldsymbol{\eta})$. Since the space $\mathbf{V}_K(\omega)$ is dense in $\mathbf{V}_M(\omega)$, it follows that $B_M(\boldsymbol{\zeta}^*,\boldsymbol{\eta}) = L(\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in \mathbf{V}_M(\omega)$.

Hence

$$\boldsymbol{\zeta}^* = \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta} \in \mathbf{V}_M(\omega)$ is the unique solution to problem $\mathcal{P}_M(\omega)$ (Theorem 8.2). Furthermore, the weak convergence

$$\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta} \text{ in } \mathbf{V}_M(\omega)$$

holds for the whole family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$.

By the inequality of Korn's type on an elliptic surface, establishing the *strong* convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $\mathbf{V}_M(\omega)$ is equivalent to establishing the convergence

$$B_M(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) \to 0,$$

which itself easily follows by letting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in the scaled Koiter equations, by noting that $B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \leq L(\boldsymbol{\zeta}_K^\varepsilon)$, and by using the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\mathbf{V}_M(\omega)$. □

Note that the convergence results of Theorem 11.3 have been improved by Lods and Mardare (1998*b*, 2000*b*), who showed that

$$\left\| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon \, \mathrm{d}x_3^\varepsilon - \boldsymbol{\zeta}_K^\varepsilon \right\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)} = O(\varepsilon^{1/5}),$$

and by Mardare (1998*b*, Theorem 5.1), who showed that

$$\|\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)} = O(\varepsilon^{1/5}).$$

Under the assumptions of Theorem 11.3, the function $\zeta_{3,K}^\varepsilon$ thus 'loses its boundary condition' as ε approaches zero. We have already remarked in Section 8.3 that, under the same assumptions, a similar 'loss of boundary condition' is shared by the average $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon \, \mathrm{d}x_3^\varepsilon$ as ε approaches zero.

We next consider *generalized membrane shells*, as defined in Section 9.1.

In the same way that in Section 9.2 we required the applied forces to be 'admissible' in order to carry out (in Theorem 9.1) the asymptotic analysis of the three-dimensional solutions, we need to assume that the applied forces enter Koiter's equations in such a way that the corresponding (scaled) linear forms are continuous with respect to the norm $|\cdot|_\omega^M$ of the 'limit' space $\mathbf{V}_M^\sharp(\omega)$, and uniformly so with respect to ε.

More specifically, we set the following definition, after Ciarlet and Lods (1996*d*) (notice the analogy with that given in Section 9.2). Applied forces are admissible for the two-dimensional Koiter equations if there exist functions $\varphi^{\alpha\beta} = \varphi^{\beta\alpha} \in L^2(\omega)$ such that, for each $\varepsilon > 0$, the right-hand side in Koiter's equations can also be written as

$$\int_\omega p^{i,\varepsilon} \eta_i \sqrt{a} \, \mathrm{d}y = \varepsilon \int_\omega \varphi^{\alpha\beta} \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a} \, \mathrm{d}y \text{ for all } \boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega).$$

As in Section 9.3, we let

$$\mathbf{V}^\sharp_M(\omega) := \text{completion of } \mathbf{V}(\omega) \text{ with respect to } |\cdot|^M_\omega,$$

where

$$\begin{aligned}\mathbf{V}(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in \mathbf{H}^1(\omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\},\\ |\boldsymbol{\eta}|^M_\omega &:= \left\{\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega}\right\}^{1/2}.\end{aligned}$$

As in Section 9.3, we restrict ourselves to generalized membrane shells 'of the first kind', since we have already noted that there is no loss of generality in doing so.

Theorem 11.4: Justification of Koiter's equations for 'generalized membrane' shells. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Consider a family of linearly elastic generalized membrane shells of the first kind, with thickness 2ε approaching zero, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, and subjected to applied forces that are admissible for both the three-dimensional equations (Section 9.2) and the two-dimensional Koiter equations, the functions $\varphi^{\alpha\beta} \in L^2(\omega)$ coinciding in addition with those found in Theorem 9.1.

For each $\varepsilon > 0$, let

$$\boldsymbol{u}^\varepsilon \in \mathbf{H}^1(\Omega^\varepsilon) \quad \text{and} \quad \boldsymbol{\zeta}^\varepsilon_K \in H^1(\omega) \times H^1(\omega) \times H^2(\omega),$$

respectively, denote the solutions to the three-dimensional and two-dimensional variational problems $\mathcal{P}(\Omega^\varepsilon)$ and $\mathcal{P}^\varepsilon_K(\omega)$. Let

$$\boldsymbol{\zeta} \in \mathbf{V}^\sharp_M(\omega)$$

denote the solution to the two-dimensional 'scaled' variational problem $\mathcal{P}^\sharp_M(\omega)$ (Theorem 9.2), a solution which is thus independent of ε. Then

$$\begin{aligned}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon \, \mathrm{d}x_3^\varepsilon &\longrightarrow \boldsymbol{\zeta} \text{ in } \mathbf{V}^\sharp_M(\omega) \text{ as } \varepsilon \to 0,\\ \boldsymbol{\zeta}^\varepsilon_K &\longrightarrow \boldsymbol{\zeta} \text{ in } \mathbf{V}^\sharp_M(\omega) \text{ as } \varepsilon \to 0.\end{aligned}$$

Proof. Under the assumptions that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ and that the applied forces are admissible in the sense of Section 9.2, the convergence

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{u}^\varepsilon \, \mathrm{d}x_3^\varepsilon = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}(\varepsilon) \, \mathrm{d}x_3 \longrightarrow \boldsymbol{\zeta} \text{ in } \mathbf{V}^\sharp_M(\omega)$$

as $\varepsilon \to 0$ was already established in Theorem 9.1.

The rest of the proof is an elaboration of Caillerie and Sanchez-Palencia (1995*a*, Theorem 4.5), who established the *weak* convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\mathbf{V}_M^\sharp(\omega)$ as $\varepsilon \to 0$.

Since the space $\mathbf{V}_K(\omega)$ is dense in the space $\mathbf{V}(\omega)$ with respect to the norm $\|\cdot\|_{1,\omega}$ and there exists c such that $|\boldsymbol{\eta}|_\omega^M \leq c\|\boldsymbol{\eta}\|_{1,\omega}$ for all $\boldsymbol{\eta} \in \mathbf{V}(\omega)$, the space $\mathbf{V}_K(\omega)$ is dense in $\mathbf{V}(\omega)$ with respect to $|\cdot|_\omega^M$ and thus the space $\mathbf{V}_M^\sharp(\omega)$ is also the completion of $\mathbf{V}_K(\omega)$ with respect to $|\cdot|_\omega^M$.

Let $B_M^\sharp$ and $L_M^\sharp$ denote the unique continuous extensions from $\mathbf{V}(\omega)$ to $\mathbf{V}_M^\sharp(\omega)$ of the bilinear and linear forms B_M and L_M defined by

$$B_M(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y,$$

$$L_M(\boldsymbol{\eta}) := \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y.$$

Since the applied forces are admissible for the two-dimensional Koiter equations, their solution $\boldsymbol{\zeta}_K^\varepsilon$ satisfies the scaled Koiter equations for a generalized membrane shell, namely,

$$B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) = L_M(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_K(\omega),$$

where

$$B_F(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y.$$

Setting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in these scaled equations then shows that the family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ is bounded in the space $\mathbf{V}_M^\sharp(\omega)$ and that the families $(\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$.

Consequently, there exists a subsequence, still denoted by $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ for convenience, and there exist $\boldsymbol{\zeta}^* \in \mathbf{V}_M^\sharp(\omega)$ and $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$ such that

$$\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^* \text{ in } \mathbf{V}_M^\sharp(\omega) \text{ and } \varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \rightharpoonup \rho_{\alpha\beta}^{-1} \text{ in } L^2(\omega).$$

Fix $\boldsymbol{\eta} \in \mathbf{V}_K(\omega)$ in the scaled Koiter equations and let $\varepsilon \to 0$; then the above weak convergence yields $B_M^\sharp(\boldsymbol{\zeta}^*, \boldsymbol{\eta}) = L_M(\boldsymbol{\eta})$. Since $\mathbf{V}_K(\omega)$ is dense in $\mathbf{V}_M^\sharp(\omega)$, it follows that $B_M^\sharp(\boldsymbol{\zeta}^*, \boldsymbol{\eta}) = L_M^\sharp(\boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in \mathbf{V}_M^\sharp(\omega)$. Hence

$$\boldsymbol{\zeta}^* = \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta} \in \mathbf{V}_M^\sharp(\omega)$ is the unique solution to the scaled problem $\mathcal{P}_M^\sharp(\omega)$ (Theorem 9.2). Furthermore, the *weak* convergence

$$\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta} \text{ in } \mathbf{V}_M^\sharp(\omega)$$

then holds for the *whole* family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$.

By definition of the norm $|\cdot|_\omega^M$ and of the bilinear form B_M and of its extension $B_M^\sharp$, establishing the *strong* convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $\mathbf{V}_M^\sharp(\omega)$ is equivalent to establishing the convergence

$$B_M^\sharp(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) \to 0,$$

which itself easily follows by letting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in the scaled Koiter equations, by noting that $B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \le L_M(\boldsymbol{\zeta}_K^\varepsilon)$, and by using the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\mathbf{V}_M^\sharp(\omega)$. $\square$

Finally, we consider *flexural shells*, as defined in Section 10.1

Theorem 11.5: Justification of Koiter's equations for 'flexural' shells. Assume that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Consider a family of linearly elastic flexural shells, with thickness 2ε approaching zero, with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and with each subjected to a boundary condition of place along a portion of its lateral face having the same set $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, and let the assumptions on the data be as in Theorem 10.1.

For each $\varepsilon > 0$, let

$$(u_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon) \text{ and } \boldsymbol{\zeta}_K^\varepsilon = (\zeta_{i,K}^\varepsilon) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega),$$

respectively, denote the solutions to the three-dimensional and two-dimensional variational problems $\mathcal{P}(\Omega^\varepsilon)$ and $\mathcal{P}_K^\varepsilon(\omega)$. Also, let

$$\boldsymbol{\zeta} = (\zeta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$$

denote the solution to the two-dimensional scaled variational problem $\mathcal{P}_F(\omega)$ (Theorem 10.2), a solution which is thus independent of ε. Then

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \, \mathrm{d}x_3^\varepsilon \longrightarrow \zeta_i \text{ in } H^1(\omega),$$
$$\zeta_{K,\alpha}^\varepsilon \longrightarrow \zeta_\alpha \text{ in } H^1(\omega) \text{ and } \zeta_{K,3}^\varepsilon \longrightarrow \zeta_3 \text{ in } H^2(\omega).$$

Proof. Under the assumptions that there exist constants $\lambda > 0$ and $\mu > 0$ and functions $f^i \in L^2(\Omega)$ independent of ε such that

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$
$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

(these are the assumptions on the data for a family of linearly elastic flexural shells) and that $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$, then

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u_i^\varepsilon \, \mathrm{d}x_3^\varepsilon = \frac{1}{2\varepsilon} \int_{-1}^{1} u_i(\varepsilon) \, \mathrm{d}x_3^\varepsilon \longrightarrow \zeta_i \text{ in } \mathbf{H}^1(\omega)$$

as $\varepsilon \to 0$ are easy corollaries to the fundamental convergence result of Theorem 10.1.

The *weak* convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ was first established by Sanchez-Palencia (1989*a*, Theorem 2.1), as a consequence of general results in perturbation theory.

We directly establish here that, in fact, the *strong* convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $\mathbf{V}_K(\omega)$ holds. Let the bilinear forms B_M and B_F and the linear form L be defined as in the proof of Theorem 11.3; in addition, let

$$\mathbf{V}_F(\omega) := \{\boldsymbol{\eta} \in \mathbf{V}_K(\omega) : \gamma_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \subset \mathbf{V}_K(\omega),$$

the space $\mathbf{V}_K(\omega)$ being equipped with the norm

$$\boldsymbol{\eta} = (\eta_i) \to \left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}.$$

By virtue of the assumptions on the applied forces, the solution $\boldsymbol{\zeta}_K^\varepsilon$ also satisfies the scaled Koiter equations for a flexural shell (it is instructive to compare them with those for an elliptic membrane shell introduced in the proof of Theorem 11.3), namely,

$$\frac{1}{\varepsilon^2} B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \boldsymbol{\eta} \in \mathbf{V}_K(\omega).$$

Letting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in these scaled equations and using the inequality of Korn's type on a general surface (Theorem 4.4) then show that the family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{V}_K(\omega)$ and that the families $\left(\frac{1}{\varepsilon}\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon)\right)_{\varepsilon>0}$ and $(\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\omega)$.

Consequently, there exists a subsequence, still denoted by $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ for convenience, and there exists $\boldsymbol{\zeta}^* \in \mathbf{V}_K(\omega)$ such that

$$\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^* \text{ in } \mathbf{V}_K(\omega) \text{ and } \gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \to 0 \text{ in } L^2(\omega).$$

The weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^*$ in $\mathbf{V}_K(\omega)$ implies the weak convergence $\gamma_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{\zeta}^*)$ in $L^2(\omega)$; hence $\gamma_{\alpha\beta}(\boldsymbol{\zeta}^*) = 0$ and thus $\boldsymbol{\zeta}^* \in \mathbf{V}_F(\omega)$. Fix $\boldsymbol{\eta} \in \mathbf{V}_F(\omega)$ in the scaled Koiter equations and let $\varepsilon \to 0$; then the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^*$ in $\mathbf{V}_K(\omega)$ yields $B_F(\boldsymbol{\zeta}^*, \boldsymbol{\eta}) = L(\boldsymbol{\eta})$. Hence

$$\boldsymbol{\zeta}^* = \boldsymbol{\zeta},$$

where $\boldsymbol{\zeta} \in \mathbf{V}_F(\omega)$ is the unique solution to the scaled problem $\mathcal{P}_F(\omega)$ (Theorem 10.2) and the *weak* convergence then holds for the *whole* family $(\boldsymbol{\zeta}^\varepsilon)_{\varepsilon>0}$.

By the inequality of Korn's type on a general surface combined with the strong convergence $\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \to 0$ in $L^2(\omega)$ and the relations $\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = 0$, establishing the *strong* convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $\mathbf{V}_K(\omega)$ is equivalent to establishing the convergence

$$B_F(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) \to 0,$$

which itself easily follows by letting $\boldsymbol{\eta} = \boldsymbol{\zeta}_K^\varepsilon$ in the scaled Koiter equations, by noting that $B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \leq L(\boldsymbol{\zeta}_K^\varepsilon)$, and by using the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\mathbf{V}_K(\omega)$. $\square$

A major conclusion emerging from Theorems 11.3, 11.4, and 11.5 is that the two-dimensional linear Koiter equations are thus justified for all kinds of shells, since, in each case, the average across the thickness of the three-dimensional solution and the solution of Koiter's equations have the same principal part, namely, in each case the solution $\boldsymbol{\zeta}$ to the corresponding two-dimensional scaled problem, as the thickness approaches zero.

By virtue of the de-scalings, which are in each case of the form $\boldsymbol{\zeta}^\varepsilon = \boldsymbol{\zeta}$ (see Sections 8.3, 9.3, and 10.3), the above asymptotic analyses also show that the solution $\boldsymbol{\zeta}_K^\varepsilon$ of Koiter's equations is 'asymptotically as good' as the solution $\boldsymbol{\zeta}^\varepsilon$ obtained by solving either the two-dimensional problem $\mathcal{P}_M^\varepsilon(\omega)$, or the two-dimensional problem $\mathcal{P}_M^{\sharp\varepsilon}(\omega)$, or the two-dimensional problem $\mathcal{P}_F^\varepsilon(\omega)$ (see Theorems 8.3, 9.3, and 10.3), according to which category the shell falls into.

Compared to these limit two-dimensional equations, Koiter's equations thus possess two outstanding advantages: not only does using Koiter's equations avoid a 'preliminary' knowledge of the category in which a given linearly elastic shell falls into, but it also avoids the mathematical or numerical difficulties inherent to each such category, briefly summarized below.

(a) If the shell is an *elliptic membrane*, no boundary condition can be imposed on the normal component ζ_3^ε of the displacement field since ζ_3^ε is 'only' in $L^2(\omega)$!

(b) If the shell is a *generalized membrane*, the solution $\boldsymbol{\zeta}^\varepsilon$ belongs to an 'abstract' completion $\mathbf{V}_M^\sharp(\omega)$; the boundary conditions on $\boldsymbol{\zeta}^\varepsilon$ may thus be quite 'exotic'!

(c) If the shell is *flexural*, the unknown $\boldsymbol{\zeta}^\varepsilon$ is subjected to the constraints $\gamma_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) = 0$ in ω, which certainly hinder its numerical approximation!

It is to be strongly emphasized that these conclusions could not be reached by an asymptotic analysis of Koiter's equations alone, for they definitely rely on an asymptotic analysis of the three-dimensional equations, namely, the content of Sections 8, 9, and 10!

Note that engineers and experts in computational mechanics often base their classification of linearly elastic shells on the relative orders of magnitudes of the 'membrane' and 'flexural' strain energies, namely,

$$\frac{\varepsilon}{2} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon) \gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \sqrt{a}\, \mathrm{d}y$$

and

$$\frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}(\boldsymbol{\zeta}_K^\varepsilon) \rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \sqrt{a}\, \mathrm{d}y,$$

found in Koiter's energy j_K^ε (Section 11.1) evaluated at a given solution $\boldsymbol{\zeta}_K^\varepsilon$, rather than on an asymptotic analysis of the three-dimensional solution as here. This approach, in which the applied forces may thus also dictate either 'membrane-dominated' or 'flexural-dominated' behaviour, has recently been given a mathematical basis by Blouza, Brezzi and Lovadina (1999).

Koiter's equations are often used for identifying and approximating boundary layers in shells; see Hakula and Pitkäranta (1995), Hakula (1997), Gerdes, Matache and Schwab (1998).

By contrast with 'boundary' layers, *'interior' layers*, that is, 'away from the lateral face', may appear inside shells with a hyperbolic middle surface. This challenging phenomenon seems again to be well modelled by Koiter's equations, as suggested by Sanchez-Palencia and Sanchez-Hubert (1998). See also Karamian (1998*a*), Leguillon, Sanchez-Hubert and Sanchez-Palencia (1999), Pitkäranta, Matache and Schwab (2000).

Koiter's equations may be adapted to the modelling of shells with periodically varying thickness, by means of a homogenization procedure; see Telega and Lewiński (1998*a*, 1998*b*), and Lewiński and Telega (2000). They may likewise be adapted to shells made of anisotropic and nonhomogeneous elastic materials, in which case additional terms in the strain energy couple the linearized change of metric and linearized change of curvature tensors; see Caillerie and Sanchez-Palencia (1995*a*), Figueiredo and Leal (1998).

11.3. Koiter's equations for shells whose middle surface has little regularity

In Section 5, we described how Blouza and Le Dret (1999) showed that the introduction of new expressions $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ and $\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})$ (reproduced below) for the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ allows us to consider more general situations, where the mapping $\boldsymbol{\theta}$ need only be in the space $W^{2,\infty}(\omega;\mathbb{R}^3)$. See also Blouza and Le Dret (2000) for further developments of this approach.

For a linearly elastic shell, simply supported along its entire boundary (boundary conditions of clamping along a portion of its boundary can be handled as well, provided they are first re-interpreted in an *ad hoc* manner), the associated 'Koiter's equations for shells whose middle surface has little regularity' accordingly take the following form. The unknown $\widetilde{\boldsymbol{\zeta}}_K^\varepsilon$, which is now the displacement field of the middle surface, satisfies the variational problem $\widetilde{\mathcal{P}}_K^\varepsilon(\omega)$:

$$\widetilde{\boldsymbol{\zeta}}_K^\varepsilon \in \widetilde{\mathbf{V}}_K^s(\omega) := \{\widetilde{\boldsymbol{\eta}} \in \mathbf{H}_0^1(\omega) : \partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_3 \in L^2(\omega)\},$$

$$\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\widetilde{\gamma}_{\sigma\tau}(\widetilde{\boldsymbol{\zeta}}_K^\varepsilon)\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\widetilde{\rho}_{\sigma\tau}(\widetilde{\boldsymbol{\zeta}}_K^\varepsilon)\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})\Big\}\sqrt{a}\,\mathrm{d}y$$
$$= \int_\omega \widetilde{\boldsymbol{p}}^\varepsilon \cdot \widetilde{\boldsymbol{\eta}}\sqrt{a}\,\mathrm{d}y$$

for all $\widetilde{\boldsymbol{\eta}} \in \widetilde{\mathbf{V}}^s_K(\omega)$, where

$$\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) := \frac{1}{2}(\partial_\beta\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\alpha + \partial_\alpha\widetilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta),$$
$$\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) := (\partial_{\alpha\beta}\widetilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\widetilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,$$

the given function $\widetilde{\boldsymbol{p}}^\varepsilon \in \mathbf{L}^2(\omega)$ accounts for the applied forces, and $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the usual contravariant components of the two-dimensional elasticity tensor of the shell.

Recall that $\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\widetilde{\boldsymbol{\rho}}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) = \rho_{\alpha\beta}(\boldsymbol{\eta})$ if $\widetilde{\boldsymbol{\eta}} = \eta_i\boldsymbol{a}^i$ is such that $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.

A proof similar to that of Theorem 11.1, now based on the inequality of Korn's type on a surface with little regularity (Theorem 5.2), then produces the following result.

Theorem 11.6: Existence and uniqueness of solutions. Let there be given a domain ω in $\mathbb{R}^2$ and an injective mapping $\boldsymbol{\theta} \in W^{2,\infty}(\omega;\mathbb{R}^3)$ such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha\boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$.

Then the associated 'Koiter's equations $\widetilde{\mathcal{P}}^\varepsilon_K(\omega)$ for a shell with little regularity' have exactly one solution, which is also the unique solution to the minimization problem:
Find $\widetilde{\boldsymbol{\zeta}}^\varepsilon_K$ such that

$$\widetilde{\boldsymbol{\zeta}}^\varepsilon_K \in \widetilde{\mathbf{V}}^s_K(\omega) \text{ and } \widetilde{j}^\varepsilon_K(\widetilde{\boldsymbol{\zeta}}^\varepsilon_K) = \inf_{\widetilde{\boldsymbol{\eta}} \in \widetilde{\mathbf{V}}^s_K(\omega)} \widetilde{j}^\varepsilon_K(\widetilde{\boldsymbol{\eta}}), \text{ where}$$

$$\widetilde{j}^\varepsilon_K(\widetilde{\boldsymbol{\eta}}) := \frac{1}{2}\int_\omega \Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\widetilde{\gamma}_{\sigma\tau}(\widetilde{\boldsymbol{\eta}})\widetilde{\gamma}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\widetilde{\rho}_{\sigma\tau}(\widetilde{\boldsymbol{\eta}})\widetilde{\rho}_{\alpha\beta}(\widetilde{\boldsymbol{\eta}})\Big\}\sqrt{a}\,\mathrm{d}y - \int_\omega \widetilde{\boldsymbol{p}}^\varepsilon \cdot \widetilde{\boldsymbol{\eta}}\sqrt{a}\,\mathrm{d}y. \qquad \square$$

It must be emphasized that, in this approach, the unknown $\widetilde{\boldsymbol{\zeta}}^\varepsilon_K$ and the fields $\widetilde{\boldsymbol{\eta}}$ are displacement fields of the middle surface, no longer recovered in general as $\widetilde{\boldsymbol{\zeta}}^\varepsilon_K = \zeta^\varepsilon_{K,i}\boldsymbol{a}^i$ or $\widetilde{\boldsymbol{\eta}} = \eta_i\boldsymbol{a}^i$ by means of their covariant components $\zeta^\varepsilon_{K,i}$ or η_i.

11.4. Budiansky–Sanders equations

Sanders (1959) and Koiter (1960) have proposed a linear shell theory akin to Koiter's, where the covariant components $\rho_{\alpha\beta}(\boldsymbol{\eta})$ of the linearized change of curvature tensor are replaced by the covariant components $\rho^{BS}_{\alpha\beta}(\boldsymbol{\eta})$ of the 'Budiansky–Sanders linearized change of curvature tensor', defined by

$$\rho^{BS}_{\alpha\beta}(\boldsymbol{\eta}) := \rho_{\alpha\beta}(\boldsymbol{\eta}) - \frac{1}{2}(b^\sigma_\alpha\gamma_{\sigma\beta}(\boldsymbol{\eta}) + b^\tau_\beta\gamma_{\tau\alpha}(\boldsymbol{\eta})).$$

The remaining terms in the equations are otherwise identical to those in

Koiter's equations. In other words, the Budiansky–Sanders equations take the following form, when they are stated as a variational problem $\mathcal{P}_{BS}^{\varepsilon}(\omega)$: Find $\boldsymbol{\zeta}^{\varepsilon} = (\zeta_i^{\varepsilon})$ such that

$$\boldsymbol{\zeta}^{\varepsilon} \in \mathbf{V}_K(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{ \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^{\varepsilon}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}^{BS}(\boldsymbol{\zeta}^{\varepsilon}) \rho_{\alpha\beta}^{BS}(\boldsymbol{\eta}) \Big\} \sqrt{a}\, \mathrm{d}y \\ = \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$.

The interest of using the modified functions $\rho_{\alpha\beta}^{BS}(\boldsymbol{\eta})$, rather than the 'genuine' functions $\rho_{\alpha\beta}(\boldsymbol{\eta})$, has been discussed at length in Budiansky and Sanders (1967) and, for this reason, the resulting theory has become known as the Budiansky–Sanders theory.

In addition, Destuynder (1985) has shown how this theory can be derived from three-dimensional linearized elasticity, again on the basis of two *a priori* assumptions, *both* of a geometrical nature, one of them being the linearized Kirchhoff–Love assumption (Section 11.1).

Theorem 11.7: Existence and uniqueness of solutions. Let the assumptions be as in Theorem 11.1. Then the associated Budiansky–Sanders equations $\mathcal{P}_{BS}^{\varepsilon}(\omega)$ have exactly one solution (which is also the unique solution to a minimization problem, the form of which should be clear).

Proof. The definition of the functions $\rho_{\alpha\beta}^{BS}(\boldsymbol{\eta})$ and the equivalence

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}^{BS}(\boldsymbol{\eta}) = 0 \text{ in } \omega \Leftrightarrow \gamma_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega$$

together imply that the proof of the existence and uniqueness of the solution to Koiter's equations (Section 4 and Theorem 11.1) extends almost *verbatim* to the Budiansky–Sanders equations. □

12. Naghdi's equations

While Koiter's equations belong to the family of Kirchhoff–Love theories, two-dimensional shell equations that rely on the notion of one-director Cosserat surfaces were proposed by P. M. Naghdi, again in the sixties. Since then, they have appealed as much as Koiter's equations to the computational mechanics community. In particular, they seem to be quite effective in the numerical simulation of shells with a 'moderately small' thickness; in this respect, see the companion article by Dominique Chapelle.

After describing the associated two-dimensional Naghdi equations for a linearly elastic shell, we briefly review in this section the existence and uniqueness theory for these equations.

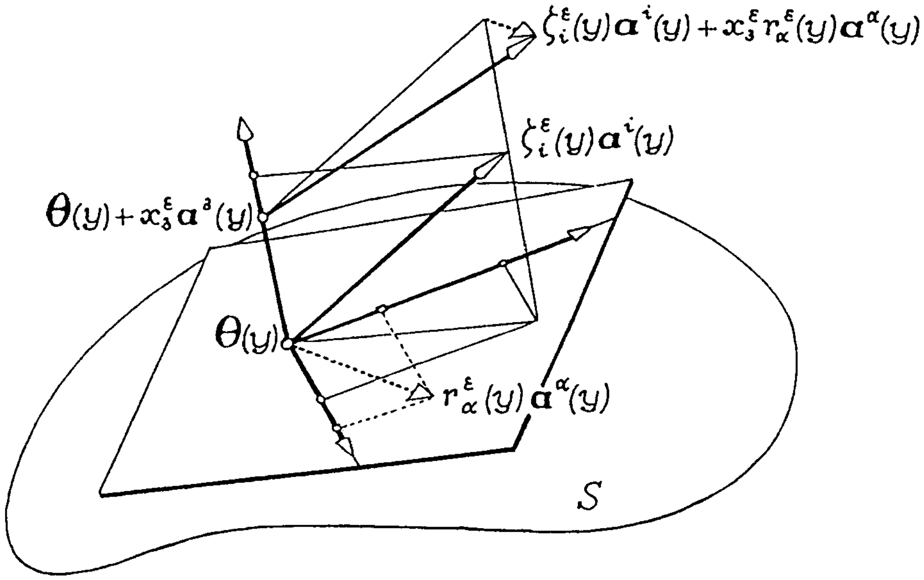

Fig. 12.1. The five unknowns in Naghdi's equations are the three covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the displacement field of the middle surface S and the two covariant components $r_\alpha^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the linearized rotation field of the unit normal vector along S; this means that, for each $y \in \overline{\omega}$, $\zeta_i^\varepsilon(y)\boldsymbol{a}^i(y) + x_3^\varepsilon r_\alpha^\varepsilon(y)\boldsymbol{a}^\alpha(y)$ is the displacement of the point $(\boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}^3(y))$ of the reference configuration of the shell

Consider as in Section 11.1 a shell with middle surface $S = \boldsymbol{\theta}(\overline{\omega})$ and thickness $2\varepsilon > 0$, constituted by a homogeneous and isotropic linear elastic material with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$, and subjected to applied body forces with contravariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$.

In Naghdi's approach (Naghdi 1963, 1972), the *a priori* assumption of a *mechanical* nature about the stresses inside the shell is the same as in Koiter's approach (Section 11.1), but the *a priori* assumption of a *geometrical* nature is different. The points situated on a line normal to S remain on a line and the lengths are unmodified along this line after the deformation has taken place as in Koiter's approach, but this line need no longer remain normal to the deformed middle surface.

In the linearized version of this approach described here, there are *five* unknowns: the three covariant components $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i$ of the middle surface S and the two covariant components $r_\alpha^\varepsilon : \overline{\omega} \to \mathbb{R}$ of the linearized rotation field $r_\alpha^\varepsilon \boldsymbol{a}^\alpha$ of the unit normal vector along S. This means that the displacement of the point $(\boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}^3(y))$ is the vector

$(\zeta_i^\varepsilon(y)\boldsymbol{a}^i(y) + x_3^\varepsilon r_\alpha^\varepsilon(y)\boldsymbol{a}^\alpha(y))$; see Figure 12.1. The surface S thus becomes a Cosserat surface, in the sense that it is endowed with the field $r_\alpha^\varepsilon \boldsymbol{a}^\alpha$, then called a director field (it is easily seen that the rotation field of the unit normal should be indeed tangential in a linearized theory).

In their weak formulation, Naghdi's equations for a linearly elastic shell consist in solving the following variational problem $\mathcal{P}_N^\varepsilon(\omega)$:
Find $(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{r}^\varepsilon) = ((\zeta_i^\varepsilon), (r_\alpha^\varepsilon))$ such that

$$(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{r}^\varepsilon) \in \mathbf{V}_N(\omega) := \{(\boldsymbol{\eta}, \boldsymbol{s}) = ((\eta_i), (s_\alpha)) \in \mathbf{H}^1(\omega) : \eta_i = s_\alpha = 0 \text{ on } \gamma_0\},$$

$$\varepsilon \int_\omega \left\{ a^{\alpha\beta\sigma\tau,\varepsilon} \gamma_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)\gamma_{\alpha\beta}(\boldsymbol{\eta}) + c\mu^\varepsilon a^{\alpha\beta}\gamma_{\alpha 3}(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{r}^\varepsilon)\gamma_{\beta 3}(\boldsymbol{\eta}, \boldsymbol{s}) \right\} \sqrt{a}\,\mathrm{d}y$$
$$+ \frac{\varepsilon^3}{3} \int_\omega a^{\alpha\beta\sigma\tau,\varepsilon} \rho_{\sigma\tau}^N(\boldsymbol{\zeta}^\varepsilon, \boldsymbol{r}^\varepsilon)\rho_{\alpha\beta}^N(\boldsymbol{\eta}, \boldsymbol{s})\sqrt{a}\,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i \sqrt{a}\,\mathrm{d}y$$

for all $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{V}_N(\omega)$ (the notation $\mathbf{H}^1(\Omega)$ standing for the space $(H^1(\omega))^5$ in the definition of the space $\mathbf{V}_N(\omega)$), where

$$\begin{aligned}
a^{\alpha\beta\sigma\tau,\varepsilon} &:= \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}),\\
\gamma_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \Gamma_{\alpha\beta}^\sigma\eta_\sigma - b_{\alpha\beta}\eta_3,\\
\gamma_{\alpha 3}(\boldsymbol{\eta}, \boldsymbol{s}) &:= \frac{1}{2}(\partial_\alpha\eta_3 + b_\alpha^\sigma\eta_\sigma + s_\alpha),\\
\rho_{\alpha\beta}^N(\boldsymbol{\eta}, \boldsymbol{s}) &:= -\frac{1}{2}(\partial_\beta s_\alpha + \partial_\alpha s_\beta) + \Gamma_{\alpha\beta}^\sigma s_\sigma - b_\alpha^\sigma b_{\sigma\beta}\eta_3\\
&\qquad + \frac{1}{2}b_\alpha^\sigma(\partial_\beta\eta_\sigma - \Gamma_{\beta\sigma}^\tau\eta_\tau) + \frac{1}{2}b_\beta^\tau(\partial_\alpha\eta_\tau - \Gamma_{\alpha\tau}^\sigma\eta_\sigma),\\
p^{i,\varepsilon} &:= \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\,\mathrm{d}x_3^\varepsilon
\end{aligned}$$

(the functions $a^{\alpha\beta}, b_{\alpha\beta}, b_\alpha^\sigma, \Gamma_{\alpha\beta}^\sigma$, and a defined as usual: see Section 4) and c is a strictly positive constant (what should be the 'best' constant seems to be an unresolved issue).

The functions $a^{\alpha\beta\sigma\tau,\varepsilon}$ are the contravariant components of the two-dimensional elasticity tensor of the shell and the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ are the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i\boldsymbol{a}^i$ of the middle surface S, as before. The 'new' functions $\gamma_{\alpha 3}(\boldsymbol{\eta}, \boldsymbol{s})$ and $\rho_{\alpha\beta}^N(\boldsymbol{\eta}, \boldsymbol{s})$ are the covariant components of the linearized transverse shear strain tensor and of the Naghdi linearized change of curvature tensor associated with displacement and linearized rotation fields $\eta_i\boldsymbol{a}^i$ and $s_\alpha\boldsymbol{a}^\alpha$ of S; for a justification of these definitions, see, *e.g.*, Bernadou (1994, Part I, Chapter 3).

The next existence and uniqueness result for the solution to the variational problem $\mathcal{P}_N^\varepsilon(\omega)$ is due to Bernadou, Ciarlet and Miara (1994, Theorem 3.1).

Theorem 12.1: Existence and uniqueness of solutions. Let ω be a domain in $\mathbb{R}^2$, let γ_0 be a subset of $\partial\omega$ with length $\gamma_0 > 0$, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \boldsymbol{\theta}$ are linearly independent at all points of $\overline{\omega}$.

Then the associated Naghdi equations $\mathcal{P}_N^\varepsilon(\omega)$ have exactly one solution (which is also the unique solution to a minimization problem, the form of which should be clear).

Sketch of proof. Let

$$|(\boldsymbol{\eta}, \boldsymbol{s})| := \left\{ \sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\boldsymbol{\eta})|^2_{0,\omega} + \sum_\alpha |\gamma_{\alpha 3}(\boldsymbol{\eta}, \boldsymbol{s})|^2_{0,\omega} + \sum_{\alpha,\beta} |\rho^N_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{s})|^2_{0,\omega} \right\}^{1/2},$$

$$\|(\boldsymbol{\eta}, \boldsymbol{s})\| := \left\{ \sum_i |\eta_i|^2_{0,\omega} + \sum_\alpha |s_\alpha|^2_{0,\omega} + |(\boldsymbol{\eta}, \boldsymbol{s})|^2 \right\}^{1/2},$$

$$\|(\boldsymbol{\eta}, \boldsymbol{s})\|_{1,\omega} := \left\{ \|\boldsymbol{\eta}\|^2_{1,\omega} + \|\boldsymbol{s}\|^2_{1,\omega} \right\}^{1/2},$$

where $\boldsymbol{\eta} = (\eta_i)$ and $\boldsymbol{s} = (s_\alpha)$.

(i) First, the Lemma of J. L. Lions, used as in Theorem 4.1, shows that there exists a constant c_0 such that

$$\|(\boldsymbol{\eta}, \boldsymbol{s})\|_{1,\omega} \le c_0 \|(\boldsymbol{\eta}, \boldsymbol{s})\|$$

for all $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{H}^1(\omega) = \big(H^1(\omega)\big)^5$.

(ii) Next, let $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{H}^1(\omega)$ be such that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\alpha 3}(\boldsymbol{\eta}, \boldsymbol{s}) = \rho^N_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{s}) = 0 \text{ in } \omega.$$

These relations imply that $\eta_3 \in H^2(\omega)$ and that $\rho^N_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{s}) = \rho_{\alpha\beta}(\boldsymbol{\eta})$ for such fields $(\boldsymbol{\eta}, \boldsymbol{s})$. Hence Theorem 4.3(a) shows that the vector field $\eta_i \boldsymbol{a}^i$ is a linearized rigid displacement of the surface $S = \boldsymbol{\theta}(\overline{\omega})$, in the sense that there exist two vectors $\widehat{\boldsymbol{c}}, \widehat{\boldsymbol{d}} \in \mathbb{R}^3$ such that

$$\eta_i(y)\boldsymbol{a}^i(y) = \widehat{\boldsymbol{c}} + \widehat{\boldsymbol{d}} \wedge \boldsymbol{\theta}(y) \text{ for all } y \in \overline{\omega}.$$

(iii) Let $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{H}^1(\omega)$ be such that

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\alpha 3}(\boldsymbol{\eta}, \boldsymbol{s}) = \rho^N_{\alpha\beta}(\boldsymbol{\eta}, \boldsymbol{s}) = 0 \text{ in } \omega, \quad \eta_i = s_\alpha = 0 \text{ on } \gamma_0,$$

where $\gamma_0 \subset \gamma$ satisfies length $\gamma_0 > 0$. Then an argument similar to that in the proof of Theorem 4.3(b) shows that $(\boldsymbol{\eta}, \boldsymbol{s}) = (\mathbf{0}, \mathbf{0})$.

(iv) A proof by contradiction as in Theorem 4.4 shows that there exists a constant c such that another inequality of Korn's type on a general surface holds (compare with that in Theorem 4.4):

$$\|(\boldsymbol{\eta}, \boldsymbol{s})\|_{1,\omega} \leq c|(\boldsymbol{\eta}, \boldsymbol{s})|$$

for all $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{V}_N(\omega)$, where

$$\mathbf{V}_N(\omega) := \{(\boldsymbol{\eta}, \boldsymbol{s}) = ((\eta_i), (s_\alpha)) \in \mathbf{H}^1(\omega) : \eta_i = s_\alpha = 0 \text{ on } \gamma_0\}.$$

(v) Finally, let $B_N^\varepsilon : \mathbf{V}_N(\omega) \times \mathbf{V}_N(\omega) \to \mathbb{R}$ denote the bilinear form defined by the left-hand side of the variational equations in problem $\mathcal{P}_N^\varepsilon(\omega)$. Then it is easily seen that there exists a constant c_N^ε such that

$$|(\boldsymbol{\eta}, \boldsymbol{s})|^2 \leq c_N^\varepsilon B_N^\varepsilon((\boldsymbol{\eta}, \boldsymbol{s}), (\boldsymbol{\eta}, \boldsymbol{s}))$$

for all $(\boldsymbol{\eta}, \boldsymbol{s}) \in \mathbf{V}_N(\omega)$. We thus conclude that the variational problem $\mathcal{P}_N^\varepsilon(\omega)$ has exactly one solution. □

Note that parts (ii) and (iii) of the above proof constitute another linearized rigid displacement lemma on a general surface, which is due to Coutris (1978).

The variational problem $\mathcal{P}_N^\varepsilon(\omega)$ is, at least formally, equivalent to a boundary value problem, which is given in Iosifescu (200x), where the *regularity* of its solution when $\gamma_0 = \gamma$ is also studied.

In the same manner that Blouza and Le Dret (1999) have generalized Theorem 11.1 to Koiter's equations for shells whose middle surface has little regularity (Theorem 11.6), Blouza (1997) has extended Theorem 12.1 to Naghdi's equations for shells whose middle surface has little regularity (the mapping $\boldsymbol{\theta}$ need only be in the space $W^{2,\infty}(\omega; \mathbb{R}^3)$).

Various asymptotic justifications of Naghdi's equations, including error estimates, are found in Lods and Mardare (1999, 2000*a*).

13. 'Shallow' shells

According to the definition justified via a formal analysis by Ciarlet and Paumier (1996) in the nonlinear case, then justified via a convergence theorem by Ciarlet and Miara (1992*a*) in the linear case, a shell is *shallow* if the deviation of its middle surface S^ε from a plane is of the order of the thickness, that is, if the surface S^ε can be written as $S^\varepsilon = \boldsymbol{\theta}^\varepsilon(\overline{\omega})$, with a mapping $\boldsymbol{\theta}^\varepsilon : \overline{\omega} \to \mathbb{R}^3$ of the form

$$\boldsymbol{\theta}^\varepsilon(y_1, y_2) = (y_1, y_2, \varepsilon\theta(y_1, y_2)) \text{ for all } (y_1, y_2) \in \overline{\omega},$$

and $\theta : \overline{\omega} \to \mathbb{R}$ is a sufficiently smooth function that is *independent of* ε; see Figure 13.1. This specific 'variation of the middle surface with ε' thus constitutes an additional assumption on the data, special to (linear and nonlinear) shallow shell theory.

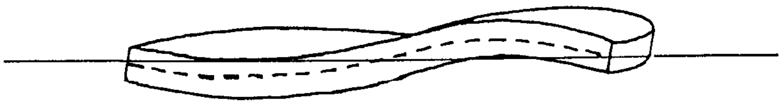

Fig. 13.1. A shell is 'shallow' if, in its reference configuration, the deviation of its middle surface from a plane is (up to an additive constant) of the order of the thickness of the shell

Like 'general' shells, linearly elastic 'shallow' shells are amenable to an asymptotic analysis (as their thickness approaches zero) that also produces 'limit' two-dimensional equations. There are, however, crucial differences between their analysis and that of 'general' shells.

First, *different scalings* are made at the *outset* of the asymptotic analysis on the *tangential* and *normal* components of the displacement field and *different assumptions* are likewise made on the *tangential* and *normal* components of the applied body force.

More specifically, another 'scaled unknown' $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ is defined in this case by letting

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon u_\alpha(\varepsilon)(x) \text{ and } u_3^\varepsilon(x^\varepsilon) = u_3(\varepsilon)(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon,$$

and it is assumed that the applied body forces are such that there exist functions $f^i \in L^2(\Omega)$ independent of ε such that

$$f^{\alpha,\varepsilon}(x^\varepsilon) = \varepsilon f^\alpha(x) \text{ and } f^{3,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^3(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

(compare with Section 7.2). Note in passing that these scalings and assumptions are identical to those made in the asymptotic analysis of linearly elastic *plates* (see Ciarlet (1997, Section 1.3)).

Making such scalings and assumptions on the data, Busse, Ciarlet and Miara (1997) have shown how two-dimensional equations of a linearly elastic shallow shell 'in curvilinear coordinates' can be given a rigorous justification by means of a convergence theorem as the thickness goes to zero. We simply list the limit equations that are found in this fashion, when they are stated as a variational problem. Let

$$\begin{aligned} b^{\alpha\beta\sigma\tau,\varepsilon} &:= \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon}\delta^{\alpha\beta}\delta^{\sigma\tau} + 2\mu^\varepsilon(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}), \\ p^{i,\varepsilon} &:= \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\,\mathrm{d}x_3^\varepsilon, \quad q^{\alpha,\varepsilon} := \int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon f^{\alpha,\varepsilon}\,\mathrm{d}x_3^\varepsilon, \\ e_{\alpha\beta}^{sh,\varepsilon}(\boldsymbol{\eta}) &:= \frac{1}{2}(\partial_\beta\eta_\alpha + \partial_\alpha\eta_\beta) - \varepsilon\eta_3\partial_{\alpha\beta}\theta \end{aligned}$$

($\delta^{\alpha\beta}$ designates the Kronecker symbol), and let $\boldsymbol{a}^{i,\varepsilon}$ designate the vectors of the contravariant bases along the middle surface S^ε (like the middle surface

S^ε, they now depend on ε). Then the 'limit', de-scaled, vector field $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$, where the functions $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ are the covariant components of the displacement field $\zeta_i^\varepsilon \boldsymbol{a}^{i,\varepsilon}$ of the middle surface S^ε, satisfies the following variational problem $\mathcal{P}_{sh}^\varepsilon(\varepsilon)$:

$$\boldsymbol{\zeta}^\varepsilon \in \mathbf{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega) : \\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

$$\int_\omega \Big\{ \varepsilon b^{\alpha\beta\sigma\tau,\varepsilon} e_{\sigma\tau}^{sh,\varepsilon}(\boldsymbol{\zeta}^\varepsilon) e_{\alpha\beta}^{sh,\varepsilon}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3} b^{\alpha\beta\sigma\tau,\varepsilon} \partial_{\sigma\tau}\zeta_3^\varepsilon \partial_{\alpha\beta}\eta_3 \Big\} \,\mathrm{d}y = \int_\omega p^{i,\varepsilon}\eta_i \,\mathrm{d}y$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}_K(\omega)$.

Another major difference thus lies in the *outcome* of the asymptotic analysis: as evidenced by the equations given above, the 'limit' variational problem simultaneously includes 'membrane' and 'flexural' terms!

More precisely, even though it is still expressed in curvilinear coordinates, the variational problem $\mathcal{P}_{sh}^\varepsilon(\omega)$ resembles more the 'limit', de-scaled, two-dimensional problem of a linearly elastic plate (see Ciarlet (1997, Section 1.7)) than that of the shell! For the contravariant components of the metric tensor usually found in the two-dimensional elasticity tensor of a shell are now replaced by Kronecker deltas, the area element along the middle surface is replaced by $\mathrm{d}y$, and finally, the components of the linearized change of metric and change of curvature tensors are replaced by the functions $e_{\alpha\beta}^{sh,\varepsilon}(\boldsymbol{\eta})$ and $\partial_{\alpha\beta}\eta^3$, where neither the Christoffel symbols nor any components of the curvature tensor of S^ε are to be found.

Problem $\mathcal{P}_{sh}^\varepsilon(\omega)$ constitutes Novozhilov's model of a shallow shell, so named after Novozhilov (1959). These equations were given a first justification by Destuynder (1980) for special geometries.

As shown by Ciarlet and Miara (1992*a*) (see also Ciarlet (1997, Chapter 3)), the two-dimensional equations 'in Cartesian coordinates' of a linearly elastic shallow shell can likewise be justified by means of an asymptotic analysis of the three-dimensional equations. As expected, and shown by Andreoiu (1999), the 'limit' displacement fields found in either curvilinear or Cartesian coordinates, though not identical vector fields, are nevertheless 'essentially the same', that is, their components agree 'to within their first orders', once they are expressed in the same basis.

The asymptotic analysis of Busse, Ciarlet and Miara (1997) has been pursued substantially further by Andreoiu, Dauge and Faou (2000) and Andreoiu and Faou (200x), who showed how to construct expansions of the scaled unknown that yield error estimates of arbitrarily high order, thus generalizing analogous results of Destuynder (1981, Corollary 7) and Dauge and Gruais (1996, 1998) for plates. Such expansions comprise a 'polynomial' part of the form $\sum_{k=0}^{p} \varepsilon^k \boldsymbol{u}^k$ (as in a formal asymptotic expansion) and a 'boundary layer' part that compensates the violation of the boundary conditions by the polynomial part.

The asymptotic analysis of the corresponding eigenvalue problem has been carried out in Cartesian coordinates by Kesavan and Sabu (1999); there is no doubt that it could be similarly carried out in curvilinear coordinates.

The exponential nature of the boundary layers that arise in linearly elastic shallow shells is analysed in Pitkäranta, Matache and Schwab (2000).

Models of multi-layered, or composite, linearly elastic shallow shells, found in particular in hulls of sailboats, have been obtained by Kail (1994) by means of the method of formal asymptotic expansions.

Other definitions of 'shallowness' have been proposed, which often make explicit reference to the curvature of the middle surface. For instance, Destuynder (1985, Section 1) considers that a shell is 'shallow' if $\eta = \varepsilon^p$ for some $p \geq 2$, where the other 'small' parameter η is the ratio of the thickness 2ε to the smallest absolute value of the radii of curvature along the middle surface, $p = 2$ corresponding to Novozhilov's model. In this direction, see also Vekua (1965), Green and Zerna (1968, p. 400), Gordeziani (1974), Dikmen (1982, p. 158), Pitkäranta, Matache and Schwab (2000).

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